UDC 517.9

The Usefulness of Cooperation in Two-Person Games with Quadratic Payoff Functions on the Rectangle

M. Aboubacar

Department of Mathematics and Computer Science
Faculty of Sciences, Abdou Moumouni University
BP: 10662 Niamey-NIGER

In this paper we study an important question for the game theory of two players, about essentiality of such games. The investigation is carried out in a particular case, in the class of games with quadratic payoff functions on a rectangle. The essentiality in two players' games means that by joining the two players in union, both players can get positive additions to guaranteed payoff. The essentiality of two players' games has not always occurred. Thus, the joining of the two players in union, in general, may be useful and sometimes (in case of absence of essentiality) useless. In applications, for example in analysis of the economic activities of firms or countries, the question of usefulness of the union acquires a lot of interest.
In the general game theory, the question of essentiality of games is given a little attention at the moment.
Apparently, this is due to the difficulty of this problem in the general case. Note that the games with quadratic payoff functions are frequently used in game theory for modeling different kinds of processes being investigated, for example, in mathematical economics.

Key words and phrases: two-person game, cooperation, usefulness, strategy, quadratic payoff functions.

1. Introduction

In game theory (see, for example [1–4]) a lot of attention is given to N person cooperative game theory. In this paper we consider the two-player game from the point of view of the usefulness of joining the players in a union, in which the choice of strategies is made concertedly, with the aim of maximizing the payoff's sum of both players.

2. Problem formulation

We consider two-person game on the rectangle $K = [p, q] \times [r, s]$ $(p < q,\ r < s)$. The payoff function of the 1st player is

$$f(x, y) = ax^2 + by^2, \tag{1}$$

where $(x, y) \in K$, a, b are arbitrary fixed non-zero numbers. The payoff function of the 2nd player is

$$g(x, y) = cx^2 + dy^2, \tag{2}$$

where $(x, y) \in K$, c, d are arbitrary fixed non-zero numbers. By selecting $x \in [p, q]$ the first player strives to maximize his payoff $f(x, y)$. By selecting $y \in [r, s]$ the second player strives to maximize his payoff $g(x, y)$. We will study the question of usefulness of joining the two players in a union, in which the choice of strategies is made concertedly, with the aim of maximizing the payoff's sum of both players. As

Received 11$^{\text{th}}$ September, 2015.
The author is grateful to Professor A. V. Arutyunov for the opportunity given to him to come to RUDN for internships at the Department of Nonlinear Analysis and Optimization, and also Professor M. S. Nikolskii for formulating the problem and helping with this article.

we know from game theory (see, for example, [1–4]) the first player can guarantee a payoff

$$\gamma_1 = \max_{x\in X} \min_{y\in Y} f(x,y), \tag{3}$$

if he selects a vector $x_0 \in X$ from the condition

$$\gamma_1 = \min_{y\in Y} f(x_0,y). \tag{4}$$

Similarly, the second player can guarantee a payoff

$$\gamma_2 = \max_{y\in Y} \min_{x\in X} g(x,y), \tag{5}$$

if he selects a vector $y_0 \in Y$ from the condition

$$\gamma_2 = \min_{x\in X} g(x,y_0). \tag{6}$$

Generally speaking, from physical point of view, payoffs $f(x,y)$, $g(x,y)$ can be measured in different physical units. We shall assume that payoffs $f(x,y)$, $g(x,y)$ are measured in the same units (in economic applications, for example, payoffs $f(x,y)$, $g(x,y)$ are usually measured in monetary units). In this kind of assumption, the value $f(x,y)+g(x,y)$ have also a physical sense. If both players are joined together in a union (coalition), then acting concertedly (i.e. choosing a pair (x,y) in $[p,q]\times[r,s]$ concertedly), they can use the quantity

$$\gamma_3 = \max_{x\in X, y\in Y} \bigl(f(x,y)+g(x,y)\bigr). \tag{7}$$

It is easy to prove that (see (3), (5))

$$\gamma_3 \geqslant \gamma_1 + \gamma_2. \tag{8}$$

If

$$\gamma_3 > \gamma_1 + \gamma_2, \tag{9}$$

the joining of the players in a union (coalition) is advantageous to both players, since the positive quantity

$$\Delta = \gamma_3 - (\gamma_1 + \gamma_2) \tag{10}$$

can be distributed in the form of positive additions to guaranteed payoffs γ_1, γ_2. How actually this distribution is reasonably to do, see, for example in [4].

3. Solution of the problem

In this paper we will study the relations (8), (9) for one class of two-person games with quadratic payoff functions on the rectangle.

Let

$$f(x,y) = f_1(x) + f_2(y), \tag{11}$$

$$g(x,y) = g_1(x) + g_2(y), \tag{12}$$

where

$$f_1(x) = ax^2, \quad f_2(y) = by^2, \quad g_1(x) = cx^2, \quad g_2(y) = dy^2, \tag{13}$$

where a, b, c, d are arbitrary fixed non-zero numbers.

In what follows, let us agree to write $\max\limits_x$ and $\max\limits_y$ instead of operations $\max\limits_{x\in[p,q]}$ and $\max\limits_{y\in[r,s]}$ respectively. Similarly, we write $\min\limits_x$ and $\min\limits_y$ instead of the operations $\min\limits_{x\in[p,q]}$

and $\min\limits_{y\in[r,s]}$ respectively. The being investigated inequality (9) can be rewritten as

$$\max_x\bigl(f_1(x)+g_1(x)\bigr)+\max_y\bigl(f_2(y)+g_2(y)\bigr)>\max_x f_1(x)+\min_y f_2(y)+ \\ +\max_y g_2(y)+\min_x g_1(x), \quad (14)$$

i.e.

$$\max_x(a+c)x^2+\max_y(b+d)y^2>\max_x ax^2+\min_x cx^2+\max_y dy^2+\min_y by^2. \quad (15)$$

We consider separately the inequalities

$$\max_{x\in[p,q]}(a+c)x^2\geqslant\max_{x\in[p,q]}ax^2+\min_{x\in[p,q]}cx^2, \quad (16)$$

$$\max_{y\in[r,s]}(b+d)y^2\geqslant\max_{y\in[r,s]}dy^2+\min_{y\in[r,s]}by^2. \quad (17)$$

Note that all the members of the inequality (16) can be directly calculated by using the following formulas:
If $\alpha>0$ and $0\in[p,q]$, then

$$\max_x\alpha x^2=\begin{cases}\alpha p^2=\alpha q^2, & \text{for}\quad |p|=|q|\\ \alpha q^2, & \text{for}\quad |p|<|q|\\ \alpha p^2, & \text{for}\quad |p|>|q|,\end{cases} \quad (18)$$

$$\min_{x\in[p,q]}\alpha x^2=0. \quad (19)$$

If $\alpha>0$ and $0\notin[p,q]$, then

$$\max_{x\in[p,q]}\alpha x^2=\begin{cases}\alpha p^2, & \text{for}\quad p<q<0\\ \alpha q^2, & \text{for}\quad q>p>0,\end{cases} \quad (20)$$

$$\min_{x\in[p,q]}\alpha x^2=\begin{cases}\alpha q^2, & \text{for}\quad p<q<0\\ \alpha p^2, & \text{for}\quad q>p>0.\end{cases} \quad (21)$$

If $\alpha<0$ and $0\in[p,q]$, then

$$\max_{x\in[p,q]}\alpha x^2=0, \quad (22)$$

$$\min_{x\in[p,q]}\alpha x^2=\begin{cases}\alpha p^2=\alpha q^2, & \text{for}\quad |p|=|q|\\ \alpha q^2, & \text{for}\quad |p|<|q|\\ \alpha p^2, & \text{for}\quad |p|>|q|.\end{cases} \quad (23)$$

If $\alpha<0$ and $0\notin[p,q]$, then

$$\max_{x\in[p,q]}\alpha x^2=\begin{cases}\alpha q^2, & \text{for}\quad p<q<0\\ \alpha p^2, & \text{for}\quad q>p>0,\end{cases} \quad (24)$$

$$\min_{x\in[p,q]}\alpha x^2=\begin{cases}\alpha p^2, & \text{for}\quad p<q<0\\ \alpha q^2, & \text{for}\quad q>p>0.\end{cases} \quad (25)$$

Considering the different types of arrangement of non-zero numbers a, c on the real straight $\mathbf{R}^1$, we reach conclusion that: the equality in (16) can only be possible on fulfilling the conditions: $a > 0$, $c < 0$, $a+c \geqslant 0$, and also on fulfilling the conditions $a < 0$, $c > 0$, $a + c \leqslant 0$. In all other cases in (16) strict inequality holds. We also similarly arrive at the following conclusion for non-zero numbers b, d: the equality in (17) can only be possible on fulfilling the conditions $b < 0$, $d > 0$, $b + d \geqslant 0$, and also on fulfilling the conditions: $b > 0$, $d < 0$, $b+d \leqslant 0$. In all other cases in (17) strict inequality holds. Thus, the inequality (16) with non-zero numbers a, c is accomplished in the strict sense on fulfilling one of the conditions:

(1) $a > 0, \quad c > 0;$
(2) $a < 0, \quad c < 0;$
(3) $a > 0, \quad c < 0, \quad a + c < 0;$
(4) $a < 0, \quad c > 0, \quad a + c > 0.$

Reasoning analogously, it can be established that inequality (17) is fulfilled in strict sense on fulfillment of one of the conditions:

(1) $b > 0, \quad d > 0;$
(2) $b < 0, \quad d < 0;$
(3) $b < 0, \quad d > 0, \quad b + d < 0;$
(4) $b > 0, \quad d < 0, \quad b + d > 0.$

The obtained facts, make it easily possible to obtain in the considered game, the sufficient conditions of fulfillment of the strict inequality (15).

4. Conclusion

In our paper we have considered a general two-person game on the rectangle. We have selected cases, when the joining of the two players in a coalition (union) is useful from the point of view of increasing the guaranteed payoff. The obtained results are of interest to the cooperative game theory. They can be used, for example, as an illustrative material in reading the course of the game theory.

References

1. N. N. Vorobyev, Game Theory for Economists Cybernetics, Nauka, Moscow, 1985, (in Russian).
2. E. Moulin, Théorie des jeux pour l'économie et la politique, Hermann, Paris, 1981.
3. L. A. Petrosyan, N. A. Zenkevich, E. A. Semina, Game Theory, High School, Moscow, 1998, (in Russian).
4. V. I. Zhukovsky, Cooperative Games with Uncertainty and Their Applications, Editorial URSS, Moscow, 1999, (in Russian).

УДК 517.9

О полезности кооперации в играх двух лиц с квадратичными функциями выигрыша на прямоугольнике

М. Абубакар

Кафедра математики информатики
Ниамейский университет им. Абду Мумуни
10662 Ниамей, Нигер

В статье исследуется важный для теории игр двух игроков вопрос о существенности таких игр. Исследование проводится в частном случае, в классе игр с квадратичными функциями выигрыша на прямоугольнике. Существенность в играх двух игроков означает, что при объединении двух игроков в союз оба игрока могут получить положительные добавки к гарантированным выигрышам. Существенность игр двух игроков имеет место далеко не всегда. Таким образом, объединение обоих игроков в союз, вообще говоря, может оказаться полезным, а иногда (в случае отсутствия существенности) и бесполезным. В приложениях, например при анализе экономической деятельности фирм или государств, вопрос о полезности союза приобретает большой интерес. В общей теории игр вопросу о существенности игр пока уделяется мало внимания. Видимо, это связано с трудностью этой проблемы в общем случае. Отметим, что игры с квадратичными функциями выигрыша часто используются в теории игр при моделировании различных процессов, исследуемых, например, в математической экономике.

Ключевые слова: игра двух игроков, кооперация, полезность, стратегия, квадратичные функции выигрыша.

Литература

1. *Воробьев Н. Н.* Теория игр для экономистов-кибернетиков. — М.: Наука, 1985.
2. *Мулен Э.* Теория игр с примерами из математической экономики. — М.: Мир, 1985.
3. *Петросян Л. А., Зенкевич Н. А., Семина Е. А.* Теория игр. — М.: Высшая школа, 1998.
4. *Жуковский В. И.* Кооперативные игры при неопределенности и их приложения. — М.: Эдиториал УРСС, 1999.