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On using Rosanes notation to specify quadratic transformations that describe a step of the Kahan difference scheme

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Abstract. Kahan's method allows approximating a dynamical system with a quadratic right-hand side using a difference scheme whose step is described by a quadratic Cremona transformation. To study such transformations, it has been proposed to use the notation of Rosanes (1870), which provides a bridge between matrix theory, which is well developed, and the theory of quadratic Cremona transformations written in projective coordinates. In this approach, the eigenvalue problem for a matrix bundle becomes the problem of finding the fundamental points of the Cremona transformation and reduces the determination of these points to a standard algebraic problem. Applying this to standard examples of dynamical systems with a quadratic right-hand side revealed a surprising observation: a common feature of the Volterra–Lotka system and the ϕ -oscillator is that in both cases, the fundamental points have coordinates in the field of fractions of the polynomial ring $\mathbb{Q}[\Delta t]$. Explicit formulas for the fundamental points in these examples are given. This property is not typical for dynamical systems with quadratic right-hand sides and is not related to the existence of algebraic integrals.

Key words and phrases: finite difference method, eigenvalue problem, Cremona transformations, birational map, eigenvalue problem

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1. Introduction

Consider the xy -plane, two equations

$$f(x, y; \hat{x}, \hat{y}) = 0, \quad g(x, y; \hat{x}, \hat{y}) = 0, \quad (1)$$

connect two pairs of variables, (x, y) and $(\hat{x}, \hat{y})$ and, in the non-degenerate case, define an algebraic correspondence on the plane: given a point (x, y) in this system, one can find several corresponding points $(\hat{x}, \hat{y})$, and vice versa.

If Eqs. (1) are linear with respect to the variables $x, y, \hat{x}, \hat{y}$, then the correspondence is called linear (affine). For example, the equations

$$x + y = \hat{x} + 2\hat{y} + 1, \quad x - y = 3\hat{y} + 4\hat{x} + 1,$$

define a linear transformation, which is usually written by expressing $\hat{x}$ and $\hat{y}$ in terms of x and y .

If Eqs. (1) are linear in both x, y and $\hat{x}, \hat{y}$, then the correspondence will still remain one-to-one, but will no longer be linear. Traditionally, this is called the quadratic Cremona transformation [1]. For example, the equations

$$x\hat{x} = 1, \quad y\hat{y} + x = 2$$



define a one-to-one transformation, in which the point (x, y) is associated with a point with coordinates

$$\hat{x} = \frac{1}{x}, \quad \hat{y} = \frac{2-x}{y},$$

and the point $(\hat{x}, \hat{y})$ is associated with a point with coordinates

$$x = \frac{1}{\hat{x}}, \quad y = \frac{2\hat{x}-1}{\hat{x}\hat{y}}.$$

Thus, a birational correspondence is obtained.

We encountered quadratic Cremona transformations while studying difference schemes for dynamical systems obtained using Kahan's method [2, 3]. In such schemes, each time step is described by a quadratic Cremona transformation. For example, the Riccati equation

$$\frac{dx}{dt} = a + bx + cx^2$$

can be approximated by the scheme

$$\hat{x} - x = \left(a + b \frac{x + \hat{x}}{2} + cx\hat{x} \right) \Delta t.$$

In this case, $\hat{x}$ is expressed fractional-linearly in terms of x and the scheme itself is symmetric under time reversal. The trick of replacing x^2 with $\hat{x}x$ was probably known in soliton theory as early as the mid-20th century, but the first publication on this scheme should be considered the report of 1993 by W. Kahan [4]. Independently of Kahan, the same technique was applied to the motion of a top by Hirota and Kimura in 1999 [5–7]; they were also the first to study the inheritance of integrals of motion by schemes of this kind. The inheritance of the energy integral in the case of a Hamiltonian system was proved in [8], and the study of integral curves in the case of a one-dimensional Hamiltonian system is presented in [9]. Following these publications, the name “Kahan's method” became established, and it was realized that the main advantage of Kahan's method is that it defines a birational mapping between the old and new positions of a dynamical system.

Obviously, there is no convenient notation for Cremona transformations. It is doubly surprising that matrix notation for quadratic Cremona transformations was proposed as early as the 1870s by J. Rosanes [10]. Unfortunately, this work went unnoticed. It is mentioned in thick reviews spanning half a century [11, 12], but no attention to Rosanes's notation is paid. It should be noted that matrix notation was just coming into use at that time, and Rosanes's presentation is not quite consistent. In particular, he fails to note the remarkable fact that fundamental points are eigenvectors of the matrix bundle describing the transformation. (Theorem 2 below). At that time, there were no matrix bundles, nor any notion of a generalized eigenvalue problem.

In this paper, we describe in detail the Rosanes notation and apply it to the study of Cremona transformations that define the steps of classical models of dynamical systems — the Volterra–Lotka system and the Hamiltonian system.

2. Quadratic Cremona transformation in Rosanes notation

In addition to the Cartesian coordinates of a point (x, y) , we will use homogeneous coordinates $(u : v : w)$, setting

$$x = \frac{u}{w}, \quad y = \frac{v}{w}.$$

When multiplying matrices, we will consider homogeneous coordinates as a column $\mathbf{u} = (u, v, w)^T$. Two matrices A and B of dimension 3×3 define a quadratic transformation

$$\hat{\mathbf{u}}^T A \mathbf{u} = 0, \quad \hat{\mathbf{u}}^T B \mathbf{u} = 0, \quad (2)$$

since these equations are linear in both $\mathbf{u}$ and $\hat{\mathbf{u}}$. It is only important that any value of $\mathbf{u}$ corresponds to a unique value of $\hat{\mathbf{u}}$, and vice versa.

For a given value $\mathbf{u}$ to correspond to a unique point $\hat{\mathbf{u}}$, it is necessary and sufficient that the equations of system (2) be linearly independent, i.e., that columns $A\mathbf{u}$ and $B\mathbf{u}$ differ by more than a multiplicative constant. In other words, a value $\mathbf{u}$ corresponds to several points $\hat{\mathbf{u}}$ if and only if there exist numbers μ, λ , one of which is nonzero, such that

$$\mu A\mathbf{u} = \lambda B\mathbf{u}. \quad (3)$$

For a given value $\hat{\mathbf{u}}$ to correspond to a single point $\mathbf{u}$, it is necessary and sufficient that the columns $\hat{\mathbf{u}}^T A\mathbf{u}$ and $\hat{\mathbf{u}}^T B\mathbf{u}$ differ by more than a multiplicative constant. In other words, the value $\hat{\mathbf{u}}$ corresponds to several points $\mathbf{u}$ if and only if there exist numbers μ, λ , one of which is nonzero, such that

$$\mu A^T \hat{\mathbf{u}} = \lambda B^T \hat{\mathbf{u}}. \quad (4)$$

From this, we can find both the conditions under which a pair of matrices defines a quadratic transformation and the conditions on the singular points of a quadratic transformation.

2.1. Conditions under which a pair of matrices defines a quadratic transformation

Condition (3) is satisfied for an arbitrary $\mathbf{u}$ if and only if the matrices are proportional to each other, that is, $\mu A = \lambda B$. The same condition holds for (4) for arbitrary $\hat{\mathbf{u}}$. This yields the first theorem.

Theorem 1. *If nonzero matrices A and B of dimension 3×3 differ by more than a multiplicative constant, then Eqs. (2) define a quadratic transformation.*

The linear space spanned by two nonzero matrices A and B , differing from each other by more than a multiplicative constant, has dimension 2. The elements of this space can be written as $\mu A - \lambda B$, where $\mu, \lambda \in k$. In operator theory, such a linear space is called a matrix bundle [13, 14]. If we take any basis A and B of the matrix bundle L , then the system (2) defines a quadratic transformation that does not change if we take another basis of this bundle.

Indeed, let A' and B' be another basis of the same matrix bundle L . Since A, B is a basis of the bundle L , the matrices A', B' are expressed linearly in terms of A, B , that is,

$$A' = \mu A - \lambda B, \quad B' = \mu' A - \lambda' B,$$

where $\mu, \dots, \lambda'$ are constants. Therefore, the equations

$$\hat{\mathbf{u}}^T A' \mathbf{u} = 0, \quad \hat{\mathbf{u}}^T B' \mathbf{u} = 0 \quad (5)$$

follow from (2). Since A', B' is a basis of the bundle L , the matrices A, B are expressed linearly in terms of A', B' and therefore Eqs. (2) follow from (5). Thus, Eqs. (2) and (5) are equivalent to each other and therefore define the same transformation of the plane.

Theorem 1 can be reformulated as follows: there is a one-to-one correspondence between bundles of 3×3 matrices and quadratic transformations.

2.2. Fundamental points of the transformation

Let us now consider the conditions (3) and (4) as conditions that indicate singular points of the transformation, i.e., the points at which the uniqueness of the transformation is violated. These points have long been called fundamental points (*punti fondamentali*) [1, n. 15].

To formulate the corresponding theorem, it is necessary to bring the terminology used into line with that generally accepted in matrix theory. Let A, B be a basis of some matrix bundle. Condition (3) can be treated as a generalized eigenvalue problem for the bundle [13, 14]. Since we are interested in the case where at least one of the quantities μ, λ is nonzero, it is natural to pass from the pair of numbers (μ, λ) to the point $(\mu : \lambda)$ on the projective straight line. Such a point will be called an eigenvalue of the pair of matrices A, B if there exists a nontrivial column $\mathbf{u}$ such that (3) for such μ, λ and $\mathbf{u}$ becomes a true equality. The column itself will be called an eigencolumn or the vector corresponding to this eigenvalue.

Theorem 2. *Quadratic transformation (2) is not unique only at those points $\mathbf{u}$ that are eigenvectors of the eigenvalue problem (3). The quadratic transformation (2) is not uniquely invertible only at those points $\hat{\mathbf{u}}$ that are eigenvectors of the eigenvalue problem (4)*

Of course, the eigenvalues of the problems (3) and (4) coincide and are the zeros of the determinant

$$\det(\mu A - \lambda B) = 0. \quad (6)$$

This is an algebraic equation of degree 3 with respect to μ/λ , so it has, generally speaking, three complex roots. If the determinant of the matrix A is zero, then among them there is a root $(\mu : \lambda) = (1 : 0)$, which we interpret as infinitely large.

If the polynomial

$$\det(\mu A - \lambda B)$$

is factored over a field k , then one of the factors is linear. Therefore, in this case, there must be at least one fundamental point. However, we cannot rule out the case where the eigenvalue is multiple and the problem has associated vectors.

Example 1. Let A be a matrix containing a Jordan block:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this case, the bundle has two eigenvectors and one associated vector. However, Eqs. (2) for this case have the form

$$\hat{v}u + \hat{w}w = 0, \quad \hat{u}u + \hat{v}v + \hat{w}w = 0$$

or in Cartesian coordinators

$$\hat{y}x + 1 = 0, \quad \hat{x}x + \hat{y}y + 1 = 0$$

and define a birational transformation of the plane.

However, in our study of the Kahan difference scheme, we did not encounter multiple eigenvalues.

3. The case when the characteristic equation factors into three different linear factors

Suppose that the left-hand side of Eq. (6) factorizes into three linear factors. Then there are three distinct eigenvalues, corresponding to three eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ of problem (3) and three distinct eigenvectors $\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3$ of problem (4). By Theorem 2, the uniqueness of the corresponding quadratic transformation is violated at exactly three distinct points in the xy -plane and three points in the $\hat{x}\hat{y}$ -plane. These points are called the fundamental points of the quadratic transformation.

3.1. Fundamental lines

Let $\mathbf{u}_i$ be an eigenvector corresponding to the eigenvalue $(\mu_i : \lambda_i)$, then in the system of equations linear with respect to $\hat{\mathbf{u}}$

$$\hat{\mathbf{u}}^T A \mathbf{u}_i = 0, \quad \hat{\mathbf{u}}^T B \mathbf{u}_i = 0,$$

one of the equations is a consequence of the other. Therefore, the point $\mathbf{u}_i$ corresponds to a straight line given by the equation

$$\hat{\mathbf{u}}^T A \mathbf{u}_i = 0. \quad (7)$$

These straight lines are called fundamental straight lines of the quadratic transformation [1, n. 15]. In this case, we say that the fundamental point $\mathbf{u}_i$ is broken by a quadratic transformation and transformed into a fundamental straight line (7).

The fundamental straight line can be described using the well-known properties of eigenvector orthogonality. Indeed, for a pair of eigenvectors $\mathbf{u}_i$ and $\hat{\mathbf{u}}_j$ corresponding to different eigenvalues, we have

$$\mu_i A \mathbf{u}_i = \lambda_i B \mathbf{u}_i,$$

whence

$$\mu_i \hat{\mathbf{u}}_j^T A \mathbf{u}_i = \lambda_i \hat{\mathbf{u}}_j^T B \mathbf{u}_i.$$

In other words,

$$\hat{\mathbf{u}}_j^T A \mathbf{u}_i : \hat{\mathbf{u}}_j^T B \mathbf{u}_i = \lambda_i : \mu_i.$$

Similarly, from

$$\mu_j \hat{\mathbf{u}}_j^T A = \lambda_j \hat{\mathbf{u}}_j^T B$$

it follows that

$$\hat{\mathbf{u}}_j^T A \mathbf{u}_i : \hat{\mathbf{u}}_j^T B \mathbf{u}_i = \lambda_j : \mu_j.$$

These equalities are possible simultaneously only when

$$\hat{\mathbf{u}}_j^T A \mathbf{u}_i = \hat{\mathbf{u}}_j^T B \mathbf{u}_i = 0.$$

Therefore, on the fundamental straight line (7), generated by the point $\mathbf{u}_i$, the fundamental points $\hat{\mathbf{u}}_j$ and $\hat{\mathbf{u}}_k$ lie, which correspond to eigenvalues different from $(\mu_i : \lambda_i)$.

Let us now consider the mapping in the opposite direction: $\hat{\mathbf{u}} \rightarrow \mathbf{u}$. The points of the fundamental straight line $\hat{\mathbf{u}}^T A \mathbf{u}_3$, passing through the fundamental points $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$, and different from the three fundamental points $\hat{\mathbf{u}}_1$, $\hat{\mathbf{u}}_2$, and $\hat{\mathbf{u}}_3$, are ordinary for the Cremona transformation under consideration and therefore map to a single well-defined point. We know this point: it is the fundamental point $\mathbf{u}_3$.

Theorem 3. *Let the equation (6) have three different roots*

$$(\mu_i : \lambda_i), \quad i = 1, 2, 3.$$

Then the quadratic transformation (2) is uniquely defined at all points except three points in the $\mathbf{u}$ plane, the eigenvalues $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ of problem (3), and three points in the $\hat{\mathbf{u}}$ plane, the eigenvalues $\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3$ of problem (4). In this case, the quadratic transformation transforms the point $\mathbf{u}_i$ into a straight line passing through $\hat{\mathbf{u}}_j$ and $\hat{\mathbf{u}}_k$ ($i \neq j, i \neq k$).

3.2. Natural coordinates

Note now that the eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ form a basis dual to the basis composed of the vectors $\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3$ [14]. Let

$$\mathbf{u} = \nu_1 \mathbf{u}_1 + \nu_2 \mathbf{u}_2 + \nu_3 \mathbf{u}_3,$$

and

$$\hat{\mathbf{u}} = \hat{\nu}_1 \hat{\mathbf{u}}_1 + \hat{\nu}_2 \hat{\mathbf{u}}_2 + \hat{\nu}_3 \hat{\mathbf{u}}_3.$$

Then, firstly,

$$\nu_i = \frac{\hat{\mathbf{u}}_i^T A \mathbf{u}}{\hat{\mathbf{u}}_i^T A \mathbf{u}_i},$$

and, secondly,

$$\hat{\mathbf{u}}^T A \mathbf{u} = \sum_{i,j} \hat{\nu}_i \nu_j \hat{\mathbf{u}}_i^T A \mathbf{u}_j = \sum_{i=1}^3 \hat{\nu}_i \nu_i \hat{\mathbf{u}}_i^T A \mathbf{u}_i.$$

We set

$$a_i = \hat{\mathbf{u}}_i^T A \mathbf{u}_i, \quad b_i = \hat{\mathbf{u}}_i^T B \mathbf{u}_i,$$

then

$$\mu_i a_i = \lambda_i b_i$$

or

$$a_i : b_i = \lambda_i : \mu_i.$$

In this case, Eqs. (2) can be rewritten as

$$\begin{cases} \hat{v}_1 v_1 a_1 + \hat{v}_2 v_2 a_2 + \hat{v}_3 v_3 a_3 = 0, \\ \hat{v}_1 v_1 b_1 + \hat{v}_2 v_2 b_2 + \hat{v}_3 v_3 b_3 = 0. \end{cases}$$

The solution of this system with respect to $\mathbf{v}$ is the vector product

$$\mathbf{v} = \det \begin{pmatrix} i & j & k \\ a_1 \hat{v}_1 & a_2 \hat{v}_2 & a_3 \hat{v}_3 \\ b_1 \hat{v}_1 & b_2 \hat{v}_2 & b_3 \hat{v}_3 \end{pmatrix},$$

and with respect to $\hat{\mathbf{v}}$ it is the vector product

$$\hat{\mathbf{v}} = \det \begin{pmatrix} i & j & k \\ a_1 v_1 & a_2 v_2 & a_3 v_3 \\ b_1 v_1 & b_2 v_2 & b_3 v_3 \end{pmatrix}.$$

For example,

$$v_1 = (a_2 b_3 - a_3 b_2) \hat{v}_2 \hat{v}_3.$$

Actually, it is now clear why the transformation is called quadratic. This notation for a quadratic transformation is used in [1, n. 17] as the definition of a quadratic transformation.

3.3. Network of straight lines

A network of straight lines in the xy -plane is defined by the equation

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0,$$

where $(c_1 : c_2 : c_3)$ are arbitrary parameters. This network is transformed into a network of lines

$$\det \begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 \hat{v}_1 & a_2 \hat{v}_2 & a_3 \hat{v}_3 \\ b_1 \hat{v}_1 & b_2 \hat{v}_2 & b_3 \hat{v}_3 \end{pmatrix} = 0.$$

This network is a network of second-order curves generated by three lines of the second order:

$$\hat{v}_1 \hat{v}_2 = 0, \quad \hat{v}_1 \hat{v}_3 = 0, \quad \hat{v}_2 \hat{v}_3 = 0.$$

Each of these lines breaks up into a pair of fundamental straight lines. For example,

$$\hat{v}_1 = 0$$

can be written in old coordinates as

$$\hat{\mathbf{u}}^T A \mathbf{u}_1 = 0.$$

This is exactly the straight line that the fundamental point $\mathbf{u}_1$ maps to. Now it is easy to see that each of these second-order lines passes through three fundamental points; therefore, the network is formed by of second-order curves passing through three fundamental points.

Initially, Cremona [15] described a quadratic transformation exactly as such a transformation of a plane that transforms a network of straight lines into a network of conic sections homologous to the original one.

3.4. Image of a line of arbitrary order

An arbitrary monomial of degree r with respect to the original Cartesian coordinates x, y transforms into a linear combination of monomials of the same degree with respect to v_1, v_2, v_3 , say,

$$v_1^{r_1} v_2^{r_2} v_3^{r_3}, \quad r_1 + r_2 + r_3 = r.$$

A quadratic transformation consists of replacing v_1 with $v_2 v_3$, and so on. Therefore, this monomial maps to

$$\hat{v}_1^{r_2+r_3} \hat{v}_2^{r_1+r_3} \hat{v}_3^{r_1+r_2}.$$

This monomial defines a line of order $2r$ consisting of fundamental straight lines. This line passes through fundamental points, e.g., through the point u_1 , and has multiplicity $r_1 + r_3 + r_1 + r_2 = r + r_1$ there. Therefore, a line of order r maps to a line of order $2r$, which passes through fundamental points of multiplicity r or higher. In particular, the linear family of all lines of order r maps to a linear family of lines of order $2r$ passing through three fundamental points of order r .

Earlier authors derived this by considering infinitely close points [1, n. 17].

4. Kahan's method

Consider a dynamical system with a quadratic right-hand side.

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y).$$

We assume that

$$f(x, y) = f_{11}x^2 + 2f_{12}xy + f_{22}y^2 + 2f_{13}x + 2f_{23}y + f_{33},$$

denote the symmetric matrix (f_{ij}) as F , and adopt a similar relation for the quadratic function g . According to Kahan's method [2, 9], this system can be approximated by a difference scheme defining a birational transformation of the plane:

$$\begin{aligned} \hat{x} - x &= (f_{11}x\hat{x} + f_{12}(x\hat{y} + \hat{y}x) + f_{22}y\hat{y} + f_{13}(x + \hat{x}) + f_{23}(y + \hat{y}) + f_{33})\Delta t, \\ \hat{y} - y &= (g_{11}x\hat{x} + \dots)\Delta t. \end{aligned} \quad (8)$$

To write this transformation in Rosanes notation, we rewrite these relations in homogeneous coordinates

$$\begin{aligned} w\hat{u} - u\hat{w} &= (f_{11}u\hat{u} + f_{12}(u\hat{v} + \hat{v}u) + \dots + f_{33}w\hat{w})\Delta t, \\ w\hat{v} - v\hat{w} &= (g_{11}u\hat{u} + \dots)\Delta t. \end{aligned}$$

We introduce

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

then in Rosanes's notation, the transformation (8) is defined by two matrices

$$A = E_1 + F\Delta t, \quad B = E_2 + G\Delta t.$$

We emphasize that F and G are symmetric matrices of the quadratic functions f and g in the generally accepted notation.

Like the original differential equations, the Kahan scheme has t -symmetry: when replacing

$$x \rightarrow \hat{x}, \quad y \rightarrow \hat{y}, \quad \Delta t \rightarrow -\Delta t$$

the scheme transforms into itself. Therefore, the inverse Cremona transformation can be calculated by replacing Δt with $-\Delta t$.

We consider two of the most well-known dynamical systems of dimension 2 with a quadratic right-hand side. For the calculations, we used the Sage system [16], and we collected the procedures used in the `rosanes` module, which we added to the FDM for Sage package [17].

4.1. Volterra–Lotka System

The Volterra–Lotka system is defined by the system of equations

$$\frac{dx}{dt} = (a - by)x, \quad \frac{dy}{dt} = (-c + dx)y.$$

The Kahan scheme for this system is given by two matrices

$$A = E_1 + \frac{\Delta t}{2} \begin{pmatrix} 0 & -b & a \\ -b & 0 & 0 \\ a & 0 & 0 \end{pmatrix}, \quad B = E_2 + \frac{\Delta t}{2} \begin{pmatrix} 0 & d & 0 \\ d & 0 & -c \\ 0 & -c & 0 \end{pmatrix}.$$

Since $\det A = \det B = 0$, there are two eigenvalues $(\mu : \lambda) = (0 : 1)$ and $(1 : 0)$. The third eigenvalue is calculated directly from (6); it is $(d : -b)$. Although all three eigenvalues are independent of Δt , the eigenvectors depend on the step.

Let us list the eigenvalues and eigenvectors.

1. The eigenvalue $(\mu : \lambda) = (1 : 0)$ corresponds to the fundamental points

$$u_1 = (0 : a\Delta t + 2 : b\Delta t), \quad \hat{u}_1 = (0 : a\Delta t - 2 : b\Delta t).$$

2. The eigenvalue $(\mu : \lambda) = (0 : 1)$ corresponds to the fundamental points

$$u_2 = (c\Delta t - 2 : 0 : d\Delta t), \quad \hat{u}_2 = (c\Delta t + 2 : 0 : d\Delta t).$$

3. The eigenvalue $(\mu : \lambda) = (d : -b)$ corresponds to the fundamental points

$$u_3 = (bc\Delta t + 2b : ad\Delta t - 2d : 0), \quad \hat{u}_3 = (bc\Delta t - 2b : ad\Delta t + 2d : 0).$$

When Δt is replaced by $-\Delta t$, the fundamental points corresponding to the same eigenvalue transform into each other:

$$u_i \rightarrow \hat{u}_i, \quad (i = 1, 2, 3).$$

It should also be noted that the points u_3 and $\hat{u}_3$ are at infinity for any Δt , while the remaining fundamental points go to infinity as $\Delta t \rightarrow 0$.

4.2. $\mathcal{H}$ -oscillator

Let us consider the $\mathcal{H}$ -oscillator, that is, a Hamiltonian system with Hamiltonian

$$h = dy^2 + ax^3 + bx^2 + cx.$$

As is known, this dynamical system is integrated exactly in Weierstrass elliptic functions [18], and Kahan's scheme imitates this property of the dynamical system [9, 19, 20]. A step of the Kahan scheme is described by a quadratic transformation with matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & d\Delta t \\ -1 & d\Delta t & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -3a\Delta t & 0 & -b\Delta t \\ 0 & 0 & 1 \\ -b\Delta t & -1 & -c\Delta t \end{pmatrix}.$$

The eigenvalue problem has three distinct roots.

1. The eigenvalue $(1 : d\Delta t)$ corresponds to the fundamental points

$$u_1 = (-2bd^2\Delta t^3 - 2d\Delta t : (b^2d^2 - 3acd^2)\Delta t^4 - 1 : 6ad^2\Delta t^3), \quad \hat{u}_1 = (0 : 1 : 0).$$

2. The eigenvalue $(1 : 0)$ corresponds to the fundamental points

$$u_2 = (d\Delta t : 1 : 0), \quad \hat{u}_2 = (-d\Delta t : 1 : 0).$$

3. The eigenvalue $(1 : -d\Delta t)$ corresponds to the fundamental points

$$u_3 = (0 : 1 : 0), \quad \hat{u}_3 = (-bd^2\Delta t^3 - 2d\Delta t : (-b^2d^2 + 3acd^2)\Delta t^4 + 1 : 6ad^2\Delta t^3).$$

It is easy to calculate the determinants of the matrices whose rows are the coordinates of the fundamental points found. Some of these determinants are equal to zero, meaning that the corresponding triple of points lies on a straight line. In this way, we discovered that the triples of points

$$u_1, \hat{u}_3, u_3 = \hat{u}_1$$

and

$$u_2, \hat{u}_2, u_3 = \hat{u}_1$$

lie on a straight line.

When Δt is replaced by $-\Delta t$, the fundamental points corresponding to the same eigenvalue transform into each other, but the rule is more sophisticated than in the case of the Volterra–Lotka system:

$$u_1 \rightarrow \hat{u}_3, \quad u_3 = \hat{u}_1, \quad u_2 \rightarrow \hat{u}_2.$$

It should be noted that there are only 5 distinct fundamental points, 3 of which are at infinity, and the remaining two go to infinity as $\Delta t \rightarrow 0$.

The energy integral of the Hamiltonian system is inherited by the Kahan difference scheme: the points of approximate solutions on the xy phase plane are aligned along lines that are the level lines of the modified Hamiltonian $h_m(x, y, \Delta t)$, which is a rational function of degree 3 [8, 9]. Since under the Cremona transformation considered, the points of the approximate solution are mapped to points of the same solution, the isolines of the modified Hamiltonian are invariant curves of the Cremona transformation. Therefore, there is a linear pencil of cubic lines invariant under the considered Cremona transformation.

On the other hand, as noted above in Section 3.4, a line of the 3rd order under the action of a quadratic transformation transforms into a line of the order $6 - r_1 - r_2 - r_3$, where r_1, r_2, r_3 are the multiplicities with which this line passes through the fundamental points u_1, u_2, u_3 . For a line of the 3rd order to transform into a line of the 3rd order, it is necessary that $r_1 + r_2 + r_3 = 3$. The simplest implementation of this equality is the assumption that the cubic line passes through these three points. Since the cubic line will also be invariant under the inverse Cremona transformation ($\mathfrak{x} \rightarrow \mathfrak{y}$), an invariant cubic line must pass through all fundamental points of the Cremona transformation. In this case, there are five of these. In other words, the five fundamental points found are base points for the pencil of isolines of the modified Hamiltonian.

We verified this statement by direct calculation. In Refs. [8, 9], an explicit expression for the modified Hamiltonian is given: h_m is a rational function of x, y , whose numerator is a polynomial of degree 3:

$$3ax^3 - 3ad^2\Delta t^2xy^2 + (3acd\Delta t^2 - b^2d\Delta t^2 + 3b)x^2 + (3d - bd^2\Delta t^2)y^2 + (3c - bcd\Delta t^2)x - c^2d\Delta t^2,$$

and the denominator is a polynomial of degree 1:

$$9ad\Delta t^2x + 3bd\Delta t^2 + 3.$$

The isolines of the Hamiltonian are cubic lines that indeed pass through all 5 fundamental points found above.

4.3. System with a cubic Hamiltonian

It is easy to find an example of a Hamiltonian system for which the roots of the characteristic equation are not rational. For example, consider

$$h = y^3 - x^3.$$

Then two of the three roots of Eq. (6) will not be elements of the field of quotients of the ring $\mathbb{Q}[\Delta t]$.

5. Discussion and results

Rosanes' notation bridges the gap between well-developed matrix theory and the theory of quadratic Cremona transformations. The eigenvalue problem for a matrix bundle turns out to be a problem of finding the fundamental points of the Cremona transformation and provides a simple method for finding these points. Applying this approach to standard examples of dynamical systems with a quadratic right-hand side yields a surprising observation: the Volterra–Lotka system and the ϕ -oscillator have in common that in both cases the fundamental points have coordinates in the field of quotients of the ring $\mathbb{Q}[\Delta t]$. This property is not typical for dynamical systems with a quadratic right-hand side and is not related, as one might think, to the existence of algebraic integrals. Nevertheless, the formulas for the fundamental points are very simple. It can be said that classical oscillators—the Volterra–Lotka system and the ϕ -oscillator—are particularly simple from the point of view of Kahan's schemes.

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Об использовании нотации Розанеса для описания квадратичных преобразований, описывающих шаг разностной схемы КаганаМ. Д. Малых^{1,2}, И. Т. Дулатов¹, А. Л. Севастьянов¹, Ин Юй³¹ Российский университет дружбы народов, ул. Миклухо-Маклая, д. 6, Москва, 117198, Российская Федерация² Объединённый институт ядерных исследований, ул. Жолио-Кюри, д. 6, Дубна, 141980, Российская Федерация³ Университет Кайли, 3, Кайюань Роуд, Кайли, 556011, Китай

Аннотация. Метод Кагана позволяет аппроксимировать динамическую систему с квадратичной правой частью разностной схемой, шаг которой описывается квадратичным преобразованием Кремоны. Для исследования таких преобразований предложено использовать нотацию Розанеса (1870), которая перекидывает мостик между теорией матриц, которая хорошо разработана, и теорий квадратичных преобразований Кремоны, записанных в проективных координатах. При этом задача на собственные значения для матричного пучка оказывается задачей об отыскании фундаментальных точек преобразования Кремоны и сводит отыскание этих точек к стандартной алгебраической задаче. Применение этого приёма к стандартным примерам динамических систем с квадратичной правой частью позволило сделать неожиданное наблюдение: систему Вольтерры–Лотки и ϕ -осциллятор объединяет то, что в обоих случаях фундаментальные точки имеют координаты в поле частных полиномиального кольца $\mathbb{Q}[\Delta t]$. Приведены явные формулы для фундаментальных точек в этих примерах. Это свойство не является типичным для динамических систем с квадратичной правой частью и не связано с существованием алгебраических интегралов.

Ключевые слова: метод конечных разностей, задача на собственные значения, преобразования Кремоны, бирациональные преобразования, задача на собственные значения