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The inverse thermography problem as the inverse potential problem for bodies of constant thickness

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Abstract. *Background* This paper examines a method for correcting thermographic images used in medical diagnostics. Mathematical processing of thermograms is performed within the framework of a mathematical model of a homogeneous, heat-conducting, semi-infinite cylinder of rectangular cross-section bounded by a general surface and containing time-independent heat sources. We assume that the density function of the source distribution corresponds to a body of constant thickness. Homogeneous conditions of the first kind are specified on the lateral faces of the cylinder, corresponding to zero temperature. Convective heat exchange with the external environment at zero temperature occurs on the surface bounding the cylindrical region. Taking blood flow into account leads to the stationary temperature distribution satisfying the inhomogeneous metaharmonic equation. *Purpose* Within the framework of the chosen model, the inverse problem of reconstructing the unknown density of heat sources from a given temperature distribution function on the surface bounding the cylinder (thermogram) is posed. This inverse problem is formulated as an ill-posed mixed problem for the metaharmonic equation with Cauchy data on the surface bounding the cylindrical region. The desired function is the right-hand side of the metaharmonic equation. *Methods* In this paper, the inverse problem using Green's formulas is reduced to an integral equation of the first kind, corresponding to the inverse potential problem for bodies of constant thickness. An approximate solution to the problem, robust to errors in the thermogram and, consequently, in the Cauchy data, is obtained using Tikhonov's regularization method. *Results* The approximate solution is constructed as a double Fourier series with a regularization factor. Convergence of the approximate solution to the problem is proven when the regularization parameter is matched to the error in the Cauchy data. *Conclusions* The approximate solution to the problem can be viewed as a corrected thermogram. Explicit formulas obtained for the approximate solution can be used to construct efficient computational algorithms.

Key words and phrases: thermogram, ill-posed problem, inverse problem, inverse potential problem, Cauchy problem for the metaharmonic equation, first-kind integral equation, Tikhonov regularization method

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1. Introduction

Digital technologies have penetrated into all branches of human activity, and one of the actual problems is to improve the quality and informativeness of the presentation of research results, in particular the quality of images obtained based on measurement data, by their mathematical (digital) processing. This applies, for example, to images obtained using thermal imaging methods with the help of a thermal imager, which records thermal electromagnetic radiation from the surface of the object under study in the infrared range. In particular, in medicine, thermal imaging has become an effective tool for early diagnosis [1–4]. The image on the thermogram, which is a map of the temperature distribution on the surface of the patient's body, makes it possible to assess functional anomalies in the state of their internal organs. At the same time, the image on the thermogram is sometimes somewhat distorted due to thermal conductivity and heat exchange processes. In the works [5, 6], a method



for correcting the image on a thermogram based on harmonic continuation (similar to problems in geophysics [7]) is proposed. In this paper, within the framework of the chosen mathematical model, the image of the density distribution function of the temperature field sources is considered as the corrected thermogram. It is proposed to obtain this function as the result of solving the corresponding inverse problem of the metaharmonic potential. The inverse potential problem is constructed based on the inverse problem for a certain mixed boundary value problem for the metaharmonic equation. The metaharmonic equation allows one to take into account heat transfer due to blood flow [8, 9]. The inverse problem under consideration is ill-posed, since small errors in the initial data (the initial thermogram) may correspond to significant errors in the solution of the inverse problem. To construct a stable approximate solution, the Tikhonov regularization method is used [10].

2. Problem statement

Let's consider a physical and then a mathematical model, within which we will set up an inverse problem.

The physical model is a homogeneous heat-conducting body in the form of a cylinder of rectangular cross-section, bounded by the surface S and containing heat sources with a density function that does not depend on time, creating a stationary temperature distribution in the body. We associate the density function of heat sources with the object under study. We assume that the lateral faces of the cylinder are maintained at zero temperature, and that there is convective heat exchange with a zero-temperature medium on the surface S , described by Newton's law, according to which the heat flux density at a point on the surface is directly proportional to the temperature difference between the inside and the outside of the body. A metaharmonic equation is used to account for blood flow.

Let us move on to the mathematical model. In an infinite cylinder with a rectangular cross-section

$$D^\infty = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, -\infty < z < \infty\} \subset \mathbb{R}^3$$

consider a cylindrical region

$$D(F, \infty) = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, F(x, y) < z < \infty\},$$

bounded by a surface

$$S = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, z = F(x, y) < H\}. \quad (1)$$

Let Γ be the set of side faces of the domain $D(F, \infty)$. In the domain $D(F, \infty)$, consider the following mixed boundary value problem for the metaharmonic equation

$$\begin{aligned} \Delta u(M) &= \rho(M), & M \in D(F, \infty), \\ \frac{\partial u}{\partial n} \Big|_S &= h(U_0 - u) \Big|_S, \\ u|_\Gamma &= 0, \\ u &\text{ is bounded when } z \rightarrow \infty. \end{aligned} \quad (2)$$

Problem (2) corresponds to a steady-state temperature distribution created by heat sources with a distribution function ρ , on the surface S there is a third boundary condition corresponding to convective heat exchange with a medium at temperature $U_0 = 0$ with a coefficient h , and the boundary Γ has a zero temperature.

We assume that the function ρ is such that the solution of problem (2) exists in

$$C^2(D(F, \infty)) \cap C^1(\overline{D(F, \infty)}).$$

In particular, the solution of problem (2) allows us to find the temperature distribution $u|_S$ on the surface S , which we call a thermogram.

Now let's set the inverse problem.

Inverse problem Let the function ρ be unknown in the model (2), but let the function

$$f = u|_S, \quad (3)$$

be given. We need to find the ρ function.

We note that the density recovery is associated with the same difficulties as the solution of the inverse potential problem [11], for which significant restrictions on the classes of uniqueness are known. Here we will consider density functions for bodies of constant thickness

$$\rho(x, y, z) = \tau(x, y)\theta(z - H)\theta(H + s - z), \quad (4)$$

belonging to the Sretensky uniqueness class [12]. In this case, we assume that the depth H and the thickness s are known, and the function $\tau(x, y)$ is the unknown quantity in the inverse problem.

In the model (2), the inverse problem is reduced to the following problem for the functions u and ρ :

$$\begin{aligned} \Delta u(M) - k^2 u(M) &= \rho(M), \quad M \in D(F, \infty), \\ u|_S &= f, \\ \frac{\partial u}{\partial n}|_S &= -hf|_S, \\ u|_F &= 0. \end{aligned} \quad (5)$$

We assume that the function f in (3), (5) is taken from the set of solutions of the problem (2) for the densities (4), so the solution of the inverse problem with respect to the function τ exists in L_2 .

We note that in the problem (5), the Cauchy conditions are specified on the surface S of the form (1), i.e., the boundary values f of the unknown function u and the values of its normal derivative are given, therefore problem (5) has a unique solution [13] with respect to u . As will be shown below, the problem with respect to τ is reduced to an inverse potential problem for the densities (4) belonging to Sretensky's uniqueness class. Problem (5), as a Cauchy problem for an elliptic equation, is not stable with respect to errors in the function f , i.e., it is ill-posed. Methods for regularizing Cauchy problems for elliptic equations are relevant even for simple domains [14, 15].

We will consider the function $\tau(x, y)$ as an adjusted thermogram.

Let's construct an explicit representation of the exact solution to problem (5).

3. Exact solution to the inverse problem

We construct the exact solution of the problem (5) following the scheme of [16–18]. Let us consider the source function $\varphi(M, P)$ of the following Dirichlet problem for the metaharmonic equation in the cylinder D^∞ :

$$\begin{aligned} \Delta_P \varphi(M, P) - k^2 \varphi(M, P) &= -\delta_{MP}, \quad M, P \in D^\infty, \\ \varphi|_{x_P=0, l_x} &= 0, \quad \varphi|_{y_P=0, l_y} = 0, \\ \varphi &\rightarrow 0 \quad \text{when } |z_P| \rightarrow \infty. \end{aligned}$$

The φ function has the form

$$\varphi(M, P) = \frac{\exp(-kr_{MP})}{4\pi r_{MP}} + W(M, P),$$

where r_{MP} is the distance between points M and P , and $W(M, P)$ is the metaharmonic function of P .

The source function can be obtained by the reflection method as a sum of point source functions with a period of $2l_x$ on x and $2l_y$ on y .

$$\varphi(M, P) = \frac{1}{4\pi} \sum_{n,m=-\infty}^{\infty} \left(\frac{\exp\{-kr_{1,nm}\}}{r_{1,nm}} - \frac{\exp\{-kr_{2,nm}\}}{r_{2,nm}} - \frac{\exp\{-kr_{3,nm}\}}{r_{3,nm}} + \frac{\exp\{-kr_{4,nm}\}}{r_{4,nm}} \right),$$

where

$$\begin{aligned} r_{1,nm} &= [(x_M - x_P + 2l_x n)^2 + (y_M - y_P + 2l_y m)^2 + (z_M - z_P)^2]^{1/2}, \\ r_{2,nm} &= [(x_M + x_P + 2l_x n)^2 + (y_M - y_P + 2l_y m)^2 + (z_M - z_P)^2]^{1/2}, \\ r_{3,nm} &= [(x_M - x_P + 2l_x n)^2 + (y_M + y_P + 2l_y m)^2 + (z_M - z_P)^2]^{1/2}, \\ r_{4,nm} &= [(x_M + x_P + 2l_x n)^2 + (y_M + y_P + 2l_y m)^2 + (z_M - z_P)^2]^{1/2}, \end{aligned}$$

so that $r_{1,00} = r_{MP}$.

Let $M \in D(-\infty, F)$, where

$$D(-\infty, F) = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, -\infty < z < F(x, y)\}.$$

Applying the second Green formula in the domain $D(F, \infty)$ to the function $u(P)$ — the solution of the problem (5) and the function $\varphi(M, P)$, we obtain

$$\begin{aligned} \int_{D(F, \infty)} [\varphi(M, P) \Delta u(P) - u(P) \Delta_P \varphi(M, P)] dv_P = \\ = \int_{\partial D(F, \infty)} \left[\frac{\partial u}{\partial n}(P) \varphi(M, P) - u(P) \frac{\partial \varphi}{\partial n_P}(M, P) \right] d\sigma_P, \quad M \in D(-\infty, F) \end{aligned}$$

or, for the metaharmonic operator

$$\begin{aligned} \int_{D(F, \infty)} [\varphi(M, P)(\Delta - k^2)u(P) - u(P)(\Delta_P - k^2)\varphi(M, P)] dv_P = \\ = \int_{\partial D(F, \infty)} \left[\frac{\partial u}{\partial n}(P) \varphi(M, P) - u(P) \frac{\partial \varphi}{\partial n_P}(M, P) \right] d\sigma_P, \quad M \in D(-\infty, F). \end{aligned}$$

Since $\varphi(M, P)$ is a metaharmonic function of P , and the function u satisfies the inhomogeneous metaharmonic equation: $(\Delta_P - k^2)u(P) = \rho(P)$, we obtain

$$\int_{D(F, \infty)} \varphi(M, P) \rho(P) dv_P = \int_{\partial D(F, \infty)} \left[\frac{\partial u}{\partial n}(P) \varphi(M, P) - u(P) \frac{\partial \varphi}{\partial n_P}(M, P) \right] d\sigma_P, \quad M \in D(-\infty, F).$$

Given the homogeneous boundary conditions for φ and u on the lateral faces Γ of the cylindrical domain $D(F, \infty)$, we obtain

$$\int_{D(F, \infty)} \varphi(M, P) \rho(P) dv_P = \int_S \left[-hf(P) \varphi(M, P) - f(P) \frac{\partial \varphi}{\partial n_P}(M, P) \right] d\sigma_P, \quad M \in D(-\infty, F). \quad (6)$$

Let us introduce the notation

$$\Phi(M) = \int_S \left[-hf(P) \varphi(M, P) - f(P) \frac{\partial \varphi}{\partial n_P}(M, P) \right] d\sigma_P, \quad M \in D(-\infty, F). \quad (7)$$

The function Φ is calculated from the known function f and can be considered as a known function. Then (6) can be written as

$$\int_{D(F, \infty)} \varphi(M, P) \rho(P) dv_P = \Phi(M), \quad M \in D(-\infty, F). \quad (8)$$

Given that according to (4) $\rho(x, y, z) = \sigma(x, y)\theta(z - H)\theta(H + s - z)$, we can write the potential in (8) as

$$\int_{D(F, \infty)} \varphi(M, P) \rho(P) dv_P = \int_0^{l_x} dx_P \int_0^{l_y} dy_P \tau(x_P, y_P) \int_H^{H+s} dz_P \varphi(M, P), \quad M \in D(-\infty, F).$$

Let $a < \min_{(x,y)} F(x, y)$ and the point M is located on the rectangle

$$\Pi(a) = \{(x, y, z) : 0 < x < l_x, 0 < y < l_y, z = a\}. \quad (9)$$

Using the representation for the source function φ when $z_P > a$

$$\varphi(M, P) = \frac{2}{l_x l_y} \sum_{n,m=1}^{\infty} \frac{e^{-k_{nm}(z_P - a)}}{k_{nm}} \sin \frac{\pi n x_M}{l_x} \sin \frac{\pi m y_M}{l_y} \sin \frac{\pi n x_P}{l_x} \sin \frac{\pi m y_P}{l_y},$$

we obtain

$$\int_{D(F, \infty)} \varphi(M, P) \rho(P) dv_P = \int_0^{l_x} dx_P \int_0^{l_y} dy_P \tau(x_P, y_P) K(M, P), \quad M \in \Pi(a). \quad (10)$$

where

$$K(M, P) = \frac{4\pi}{l_x l_y} \sum_{n,m=1}^{\infty} K_{nm} \sin \frac{\pi n x_M}{l_x} \sin \frac{\pi m y_M}{l_y} \sin \frac{\pi n x_P}{l_x} \sin \frac{\pi m y_P}{l_y}. \quad (11)$$

$$K_{nm} = \exp \left\{ -k_{nm} \left(H + \frac{s}{2} - a \right) \right\} \frac{\sinh \frac{s}{2}}{k_{nm}^2}, \quad k_{nm} = \pi \left(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2} + k^2 \right)^{1/2}. \quad (12)$$

Then, from (10) and (8), we obtain an integral equation of the first kind

$$\int_{\Pi(H)} K(M, P) \tau(P) dx_P dy_P = \Phi(M), \quad M \in \Pi(a). \quad (13)$$

From equation (13), taking into account the expansion (11), when $z_M = a$, we obtain a relation between the Fourier coefficients of the unique solution τ and the Fourier coefficients of the right-hand side

$$\tilde{\tau}_{nm} K_{nm} = \tilde{\Phi}_{nm}(a), \quad (14)$$

where $\tilde{\Phi}_{nm}(a)$ are the Fourier coefficients of the function $\Phi(M)|_{M \in \Pi(a)}$:

$$\tilde{\Phi}_{nm}(a) = \frac{4}{l_x l_y} \int_{\Pi(a)} \Phi(x, y, a) \sin \frac{\pi n x}{l_x} \sin \frac{\pi m y}{l_y} dx dy.$$

Let us note that formula (14) characterizes the decrease of the Fourier coefficients $\tilde{\Phi}_{nm}(a)$ with increasing n and m if the function f is such that it ensures the existence of a solution to the problem (5) and, consequently, of the function τ . Expressing the Fourier coefficients $\tilde{\tau}_{nm}$ from (14), we obtain

$$\tau(x, y) = \sum_{n,m=1}^{\infty} \tilde{\Phi}_{nm}(a) K_{nm}^{-1} \sin \frac{\pi n x}{l_x} \sin \frac{\pi m y}{l_y}. \quad (15)$$

The series (15) converges in L_2 if the solution of problem (5) with respect to τ exists for the given function f .

Formula (15), where the function Φ is of the form (7), gives an explicit expression for solving the inverse problem.

4. Approximate solution to the problem

Let the function f in problem (5) be given with an error, that is, instead of the function f , the function f^δ is given, such that

$$\|f^\delta - f\|_{L_2(S)} \leq \delta.$$

We construct an approximate solution of the ill-posed problem (5) that converges to the exact solution when $\delta \rightarrow 0$. The function Φ of the form (7) in this case can be obtained approximately:

$$\Phi^\delta(M) = \int_S \left[-hf^\delta(P)\varphi(M, P) - f^\delta(P)\frac{\partial\varphi}{\partial n_P}(M, P) \right] d\sigma_P. \quad (16)$$

Applying the Cauchy-Bunyakovsky inequality to the difference of the functions (16) and (7) for $M \in \Pi(a)$, $a < \min_{(x,y)} F(x, y)$, we obtain an estimate for the right-hand side of the integral equation (13)

$$\begin{aligned} |\Phi^\delta(M) - \Phi(M)|_{M \in \Pi(a)} &\leq h \max_{M \in \Pi(a)} \left(\int_S \varphi^2(M, P) d\sigma_P \right)^{1/2} \|f^\delta - f\|_{L_2(S)} + \\ &+ \max_{M \in \Pi(a)} \left(\int_S \left[\frac{\partial\varphi}{\partial n_P}(M, P) \right]^2 d\sigma_P \right)^{1/2} \|f^\delta - f\|_{L_2(S)} \leq C\delta. \end{aligned} \quad (17)$$

As an approximate solution to equation (13), we will consider the extremal of the Tikhonov functional [10] with a zero-order stabilizer

$$M^\alpha[w] = \|Kw - \Phi^\delta\|_{L_2(\Pi(a))}^2 + \alpha \|w\|_{L_2(\Pi(H))}^2, \quad \alpha > 0, \quad (18)$$

where $\Pi(a)$ and $\Pi(H)$ are the domains of the form (9).

The extremal can be obtained as a solution of the Euler equation for the functional (18)

$$K^*Kw + \alpha w = K^*\Phi^\delta(a),$$

which in the Fourier coefficients of the function w has the form

$$K_{nm}^2 \tilde{w}_{nm} + \alpha \tilde{w}_{nm} = K_{nm} \tilde{\Phi}_{nm}^\delta(a),$$

where K_{nm} has the form (12) and

$$\tilde{\Phi}_{nm}^\delta(a) = \frac{4}{l_x l_y} \int_{\Pi(a)} \Phi^\delta(x, y, a) \sin \frac{\pi n x}{l_x} \sin \frac{\pi m y}{l_y} dx dy$$

— Fourier coefficients of the function $\Phi^\delta(M)|_{M \in \Pi(a)}$.

Solving the equation for the Fourier coefficients of the extremal, we find the approximation w_α^δ to the function τ in the domain $D(-\infty, H)$:

$$w_\alpha^\delta(x, y) = \sum_{n,m=1}^{\infty} \frac{\tilde{\Phi}_{nm}^\delta(a) K_{nm}^{-1}}{1 + \alpha K_{nm}^{-2}} \sin \frac{\pi n x}{l_x} \sin \frac{\pi m y}{l_y}. \quad (19)$$

We note that the terms of the series (19) differ from the terms of the series (15) by the regularizing factor $(1 + \alpha K_{nm}^{-2})^{-1}$, which ensures the convergence of the series.

Theorem 1. *Let the solution of problem (5) exist. Then for any $\alpha = \alpha(\delta) > 0$ such that $\alpha(\delta) \rightarrow 0$ and $\delta/\sqrt{\alpha(\delta)} \rightarrow 0$ as $\delta \rightarrow 0$, the function $w_{\alpha(\delta)}^\delta$ of the form (19) converges as $\delta \rightarrow 0$ to the exact solution τ in L_2 .*

Proof. Let us estimate the difference

$$\|w_\alpha^\delta - \tau\| \leq \|w_\alpha^\delta - w_\alpha\| + \|w_\alpha - \tau\|, \quad (20)$$

where w_α is a function of the form (19) with an exact function f

$$w_\alpha(M) = \sum_{n,m=1}^{\infty} \frac{\tilde{\Phi}_{nm}(a)K_{nm}^{-1}}{1 + \alpha K_{nm}^{-2}} \sin \frac{\pi n x_M}{l_x} \sin \frac{\pi m y_M}{l_y}.$$

Let us estimate the difference $w_\alpha^\delta - w_\alpha$ in (20) using the estimate (17)

$$\begin{aligned} \|w_\alpha^\delta(M) - w_\alpha(M)\| &= \left| \frac{4}{l_x l_y} \sum_{n,m=1}^{\infty} \left(\frac{K_{nm}^{-1}}{1 + \alpha K_{nm}^{-2}} \right)^2 |\tilde{\Phi}_{nm}^\delta(a) - \tilde{\Phi}_{nm}(a)|^2 \right|^{1/2} \leq \\ &\leq C_1 \max_x \left[\frac{e^x}{1 + \alpha e^{2x}} \right] \|\Phi^\delta(a) - \Phi(a)\| \leq C_1 \frac{\delta}{\sqrt{2\alpha}}. \end{aligned} \quad (21)$$

Let us estimate the difference $w_\alpha - \tau$ in (20)

$$\|w_\alpha - \tau\| \leq \left| \frac{4}{l_x l_y} \sum_{n,m=1}^{\infty} \left(\frac{K_{nm}^{-2}}{1 + \alpha K_{nm}^{-2}} \right)^2 |K_{nm} \tilde{\Phi}_{nm}(a)|^2 \right|^{1/2} = \left| \frac{4}{l_x l_y} \sum_{n,m=1}^{\infty} \left(\frac{K_{nm}^{-2}}{1 + \alpha K_{nm}^{-2}} \right)^2 \tilde{\tau}_{nm}^2 \right|^{1/2}.$$

Since the series depending on the parameter α is majorized by a convergent numerical series with coefficients $\tilde{\tau}_{nm}^2$, a limit transition in α is possible, and thus

$$\|w_\alpha - \tau\| \rightarrow 0 \quad \text{при} \quad \alpha \rightarrow 0. \quad (22)$$

From (20), (21), and (22), and the conditions of the theorem, it follows that

$$\|w_{\alpha(\delta)}^\delta - \tau\| \rightarrow 0 \quad \text{при} \quad \delta \rightarrow 0.$$

5. Results and Discussion

The proven theorem allows us to consider the series (19) as an explicit formula for calculating an approximate solution to problem (5), robust to an error in specifying the thermogram f . Note that for a flat surface S , calculating the integral (16) is significantly simplified. When specifying a non-flat surface S approximately, the results of works [19, 20] on calculating the normal to an approximately specified surface can be used to perform calculations using formula (16).

Conclusion

A stable solution to the inverse problem (5) can be used for mathematical processing of thermograms, particularly in medicine [1], for the purpose of thermal image correction. A thermogram obtained with a thermal imager conveys with varying accuracy the structure of heat sources within the body. Refinement of the thermogram image can be performed within the framework of problem (5). In this case, the function f is compared to the original thermogram, and the function τ can be considered as the result of thermogram processing. Since the function τ represents the intensity distribution of heat sources on the rectangle $\Pi(H)$, one can expect a more accurate reproduction of their image in the calculated thermogram τ compared to the original thermogram f .

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References

- [1] G. R. Ivanitskii, "Thermovision in medicine," Russian, *Vestnik RAN*, vol. 76, no. 1, pp. 44–53, 2006.
- [2] E. F. J. Ring, "Progress in the measurement of human body temperature," *IEEE Engineering in Medicine and Biology Magazine*, vol. 17, no. 4, pp. 19–24, 1998. DOI: 10.1109/51.687959
- [3] E. Y. K. Ng and N. M. Sudarshan, "Numerical computation as a tool to aid thermographic interpretation," *Journal of Medical Engineering and Technology*, vol. 25, no. 2, pp. 53–60, 2001. DOI: 10.1080/03091900110043621
- [4] B. F. Jones and P. Plassmann, "Digital infrared thermal imaging of human skin," *IEEE Eng. in Med. Biol. Mag.*, vol. 21, no. 6, pp. 41–48, 2002. DOI: 10.1109/memb.2002.1175137
- [5] E. B. Laneev, N. Y. Chernikova, and O. Baaj, "Application of the minimum principle of a Tikhonov smoothing functional in the problem of processing thermographic data," *Advances in Systems Science and Applications*, vol. 21, no. 1, pp. 139–149, 2021. DOI: 10.25728/assa.2021.21.1.1055
- [6] E. B. Laneev and A. V. Klimishin, "On an approximate solution to an ill-posed mixed boundary value problem for the Laplace equation in a cylindrical domain with homogeneous conditions of the second kind on the lateral surface of the cylinder," Russian, *Russian Universities Reports. Mathematics*, vol. 29, no. 146, pp. 164–175, 2024. DOI: 10.20310/2686-9667-2024-29-146-164-175
- [7] A. N. Tikhonov, V. B. Glasko, O. K. Litvinenko, and V. R. Melihov, "On the continuation of the potential towards disturbing masses based on the regularization method," Russian, *Vestnik RAN*, no. 1, pp. 30–48, 1968.
- [8] H. Pennes, "Analysis of tissue and arterial blood temperature in the resting human forearm," *J. Appl. Physiol.*, no. 1, pp. 93–122, 1948.
- [9] J. Agnelli, A. Barrea, and C. Turner, "Tumor location and parameter estimation by thermography," *Mathematical and Computer Modelling*, vol. 53, pp. 1527–1534, 2011. DOI: 10.1016/j.mcm.2010.04.003
- [10] A. N. Tikhonov and V. Y. Arsenin, *Methods for solving ill-posed problems*, Russian. Moscow: Nauka, 1979.
- [11] A. I. Prilepko, "Inverse problems of potential theory," *Math. Notes*, vol. 14, no. 5, pp. 990–996, 1973. DOI: 10.1007/BF01462264
- [12] L. N. Sretenskiy, "On the uniqueness of determining the shape of an attractive body by the values of its external potential," Russian, *Doklady RAN*, vol. 99, no. 1, pp. 21–22, 1954.
- [13] E. M. Landis, "Some problems of the qualitative theory of second order elliptic equations (case of several independent variables)," *Russian Math. Surveys*, vol. 18, no. 1, pp. 1–62, 1963. DOI: 10.1070/RM1963v018n01ABEH004124
- [14] S. B. Sorokin, "An Efficient Direct Method for Numerically Solving the Cauchy Problem for Laplace's Equation," *Num. Anal. Appl.*, vol. 12, no. 1, pp. 87–103, 2019. DOI: 10.15372/SJNM20190107
- [15] M. Joachimiak, "Choice of the regularization parameter for the Cauchy problem for the Laplace equation," *International Journal of Numerical Methods for Heat and Fluid Flow*, vol. 30, no. 10, pp. 4475–4492, 2020. DOI: 10.1108/HFF-10-2019-0730
- [16] E. Laneev, "Construction of a Carleman function based on the Tikhonov regularization method in an ill-posed problem for the Laplace equation," *Differential Equations*, vol. 54, no. 4, pp. 476–485, 2018. DOI: 10.1134/S0012266118040055
- [17] E. Laneev and C. N.Yu., "Minimum Principle for the Tikhonov Functional in the Problem of Stable Continuation of a Potential Field from a Surface," *Differential Equations*, vol. 59, no. 6, pp. 752–762, 2023. DOI: 10.1134/S001226612306006X

- [18] E. Laneev and E. Ponomarenko, "On a linear inverse potential problem with approximate data on the potential field on an approximately given surface," *Eurasian Mathematical Journal*, vol. 14, no. 1, pp. 57–70, 2023. DOI: 10.32523/2077-9879-2023-14-1-55-70
- [19] O. Baaj, N. Y. Chernikova, and E. Laneev, "Correction of Thermographic Images Based on the Minimization Method of Tikhonov Functional," *Yugoslav Journal of Operations Research*, vol. 32, no. 3, pp. 407–424, 2022. DOI: 10.2298/YJOR211015026B
- [20] E. B. Laneev and A. V. Klimishin, "On the stable approximate solution of the ill-posed boundary value problem for the Laplace equation with homogeneous conditions of the second kind on the edges at inaccurate data on the approximated boundary," *Discrete and Continuous Models and Applied Computational Science*, vol. 1, no. 33, pp. 57–73, 2025. DOI: 10.22363/2658-4670-2023-31-3-228-241

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Обратная задача термографии как обратная задача потенциала для тел постоянной толщины

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Аннотация. *Предпосылки* В работе рассматривается метод коррекции термографических изображений, используемых в медицинской диагностике. Математическая обработка термограмм производится в рамках математической модели однородного теплопроводящего полубесконечного цилиндра прямоугольного сечения, ограниченного поверхностью общего вида и содержащего источники тепла, не зависящие от времени. Мы считаем, что функция плотности распределения источников соответствует телу постоянной толщины. На боковых гранях цилиндра заданы однородные условия первого рода, что соответствует нулевой температуре. На поверхности, ограничивающей цилиндрическую область имеет место конвективный теплообмен со внешней средой нулевой температуры. Учет кровотока приводит к тому, что стационарное распределение температуры удовлетворяет неоднородному метагармоническому уравнению. *Цель* В рамках выбранной модели поставлена обратная задача восстановления неизвестной плотности источников тепла по заданной функции распределения температуры на поверхности, ограничивающей цилиндр (термограмме). Эта обратная задача формулируется как некорректно поставленная смешанная задача для метагармонического уравнения с данными Коши на поверхности, ограничивающей цилиндрическую область. Искомая функция – правая часть метагармонического уравнения. *Методы* В работе обратная задача с использованием формул Грина сведена к интегральному уравнению первого рода, соответствующему обратной задаче потенциала для тел постоянной толщины. Приближенное решение поставленной задачи, устойчивое по отношению к погрешности в термограмме и, следовательно, в данных Коши, получено на основе метода регуляризации Тихонова. *Результаты* Приближенное решение построено в виде двойного ряда Фурье с регуляризующим множителем. Доказана сходимость приближенного решения поставленной задачи при согласовании параметра регуляризации с погрешностью в данных Коши. *Выводы* Приближенное решение задачи может рассматриваться в качестве скорректированной термограммы.

Явные формулы, полученные для приближенного решения, могут быть использованы для построения эффективных вычислительных алгоритмов.

Ключевые слова: термограмма, некорректная задача, обратная задача, обратная задача потенциала, задача Коши для метагармонического уравнения, интегральное уравнение первого рода, метод регуляризации Тихонова