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On the classical Volterra spectral equation

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Abstract. We study the characterizations of the classical Volterra operator based on a simple relation between its singular numbers and the eigenvalues of its imaginary part on $L^2[0,1]$.

Key words and phrases: Volterra operator, trace operator, operator norm, eigenvalue

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1. Introduction

Let H be a complex Hilbert space equipped with the inner product $(\cdot,\cdot)$ and the induced norm $\|\cdot\|$. Denote by $B(H)$ the Banach algebra of bounded linear operators acting on H with the operator norm

$$\|A\| = \sup_{\|x\|=1}\{\|Ax\| : x \in H\}, A \in B(H).$$

The integral operator A on $L^2[0,1]$ defined as

$$(Af)(x) = \int_0^1 k(x,s)f(s)ds$$

with some kernel $k(t,s) \in L^2([0,1]^2)$. Then A is a Hilbert–Schmidt operator.

Recall that for an operator A the spectrum

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{is not invertible}\}$$

is a nonempty compact subset of the complex plane.

For any compact linear operator A in a Hilbert space H the singular numbers $s_k(A)$ are the distances from A to the set of all operators of rank less than or equal to $k-1$ for $k \geqslant 1$. Their squares are the eigenvalues of the compact self-adjoint nonnegative operator A^*A counted according to their multiplicities. (see e.g. [1]) In particular, $s_1(A) = \|A\|$.

Denote by V the Volterra operator

$$(Vf)(x) = \int_0^x f(s)ds$$

and its adjoint

$$(V^* f)(x) = \int_x^1 f(s)ds$$

on $L^2[0,1]$.

The Halmos (see [2]) calculation yields (see [3]):

$$s_k(V) = \frac{2}{(2k-1)\pi}$$

for all $k \geqslant 1$, in particular

$$\|V\| = \frac{2}{\pi}.$$

Note that

$$\sum_{k=0}^{\infty} s_k^2 = \frac{1}{2}.$$

If $A \in B(H)$ and $\{\lambda_k\}$ are the eigenvalues of A, then

$$tr(A) = \sum_k \lambda_k.$$

It is well known that V is quasi-nilpotent, compact and even a Hilbert–Schmidt operator. Due to its excellent properties, many scholars have studied Volterra operator pencils on Hilbert spaces and formulated many interesting results (see e.g. [4–13]).

The set $M_\alpha = \{f \in L^2[0,1] : f(x) = 0, 0 \leqslant x \leqslant \alpha\}$ is an invariant subspace for V, for $0 \leqslant \alpha \leqslant 1$ (see [14, 15]). Recall that

$$\mathrm{Re}\, V = \frac{1}{2}(V + V^*) = \frac{1}{2}P,$$

where P is the orthogonal projection onto the constant functions.

It is easy to see that the nonzero eigenvalues of the imaginary part

$$\mathrm{Im}\, V = \frac{1}{2i}(V - V^*)$$

are

$$\pm\frac{1}{(2k-1)\pi}$$

for $k \in \mathbb{N}$, each having multiplicity 1 (see [15]). Then we have the spectral equation

$$2\sigma(\mathrm{Im}\, V) = \sigma((V^*V)^{\frac{1}{2}}) \cup (-\sigma((V^*V)^{\frac{1}{2}})),$$

where σ denotes the spectrum (see [16]).

Recall that for VV^* the orthonormal basis of $L^2[0,1]$ consisting of eigenfunctions

$$f_k(x) = \sqrt{2}\cos\left(\frac{2k+1}{2}\pi x\right)$$

corresponding to eigenvalues

$$\lambda_k = \frac{4}{(2k+1)^2\pi^2}$$

for $n \geqslant 0$ (see [17]).

We obtain the characterizations of the Volterra operator on $L^2[0,1]$.

2. The Results

Theorem 3. *Suppose that A is a Hilbert–Schmidt operator on $L^2[0,1]$ such that*

$$A + A^* = Q$$

is a rank-one projection, quasinilpotent and

$$2\sigma(Im\,A) = \sigma((A^*A)^{\frac{1}{2}}) \cup (-\sigma((A^*A)^{\frac{1}{2}})).$$

Then $A = V$.

Proof. If A is quasinilpotent and rank-one projection (3) and the spectra on both sides in (3) are countable set, then A is a compact operator (see [18, 19]). On the other hand, we observe that 0 is not an eigenvalue of A, and that A and A^* have no proper common invariant subspace. Also A is rank-one projection and Im A $\geqslant 0$ then A is unitary equivalent to the Volterra operator (see [20]).

Suppose that the eigenvalues of Im A and the singular numbers $\{s_k\}_{k\in\mathbb{N}}$ of A have multiplicity 1. From (3), we have

$$\sum_{k=1}^{\infty} s_k^2 = tr(A^*A) = tr\Big((\frac{1}{2}Q - i\text{Im A})(\frac{1}{2}Q + i\text{Im A})\Big) =$$

$$= \frac{1}{4}tr(Q) + \frac{i}{2}tr([Q,\text{Im A}]) + tr((\text{Im A})^2) =$$

$$= \frac{1}{4} + 2\sum_{k=1}^{\infty}\left(\frac{s_k}{2}\right)^2 = \frac{1}{4} + \frac{1}{2}\sum_{k=1}^{\infty} s_k^2.$$

Hence,

$$\sum_{k=0}^{\infty} s_k^2 = \frac{1}{2}.$$

Therefore $A = V$. This completes the proof. □

Conclusions

We have seen that characterizations technique for the Volterra operator is based on some relation between its singular numbers and the eigenvalues of its imaginary part on $L^2[0,1]$.

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О классическом спектральном уравнении Вольтерры

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Аннотация. Мы изучаем характеристики классического оператора Вольтерры на основе простого соотношения между его сингулярными числами и собственными значениями его мнимой части на $L^2[0,1]$.

Ключевые слова: Оператор Вольтерры, оператор следа, норма оператора, собственное значение