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On quadratic transformations of the Fano plane

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Abstract. The importance of studying quadratic Cremona transformations over algebraically non-closed fields for the theory of Kahan difference schemes for dynamical systems with a quadratic right-hand side is discussed. Cremona transformations of the projective plane over a Galois field of size 2, i.e., the Fano plane, are considered. The notation proposed by J. Rosanes is used to describe quadratic Cremona transformations. The relationship between quadratic transformations and matrix pencils is described. Quadratic transformations without singular points are called regular. The Sage system is used to implement procedures that convert a quadratic transformation into a permutation of 7 points of the Fano plane (an element of the symmetric group S_7) and a permutation of 7 points of the plane into a quadratic transformation. By enumerating all quadratic transformations, it is proved that regular quadratic transformations generate the entire permutation group of the Fano plane. This theorem is analogous to Noether's theorem over an algebraically non-closed field. It is proved that regular quadratic Cremona transformations have even order, and a description of the corresponding permutations of the group S_7 is given. It is shown that a permutation always corresponds to some quadratic Cremona transformation, but this transformation is not always regular. The resulting quadratic transformations can be supplemented by a rule that resolves ambiguities at fundamental points. An example of a quadratic transformation for which such resolution is impossible is given. This leads to a natural classification of quadratic transformations of the Fano plane.

Key words and phrases: projective plane, Galois fields, Cremona transformations, birational map, Kahan's method

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1. Introduction

We encountered Cremona transforms while studying difference schemes for dynamical systems [1] obtained using the Kahan method [2], developed in the works of Sanz-Serna [3], Suris [4–6], and McLachlan [7]. In such schemes, each time step is described by a quadratic Cremona transformation. Although the construction itself appears quite classical, the few known results prove of little use. The point is that the coefficients of difference schemes typically belong to algebraically open fields, for example, the field of fractions of the ring $\mathbb{Q}[\Delta t]$, where Δt is the time step (symbolic variable), or even to the field $\mathbb{Q}$. Moreover, an approximate solution found for a fixed rational step is an infinite

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sequence of rational numbers. Therefore, for these studies, it would be desirable to have a theory of quadratic Cremona transformations over algebraically open fields. However, research on Cremona transformations, both new and classical, is usually conducted over an algebraically closed field [8, 9].

In the case of algebraically open fields, there is one case that can be studied completely and without much effort. This is the case of finite fields. In this article, we consider quadratic transformations of the projective plane on a Galois field of size 2.

2. Quadratic transformations of the plane

Let us recall the basic definitions from classical algebraic geometry. Two equations

$$f(x, y; \hat{x}, \hat{y}) = 0, \quad g(x, y; \hat{x}, \hat{y}) = 0, \quad (1)$$

connect two pairs of variables, (x, y) and $(\hat{x}, \hat{y})$, and in the non-degenerate case define an algebraic correspondence on the plane P_2 : given an initial point (x, y) from this system, one can find several corresponding endpoints $(\hat{x}, \hat{y})$, and vice versa. If $\hat{x}, \hat{y}$ can be expressed rationally in terms of x, y , then one speaks of a rational transformation of the plane [10, no. 14], and if, in addition, if x, y can also be expressed rationally in terms of $\hat{x}, \hat{y}$, then we speak of a birational transformation (birational map) of the plane or the Cremona transformation [10, no. 16].

In particular, if the equations (1) are linear in both x, y and $\hat{x}, \hat{y}$, then the correspondence will be one-to-one, and therefore x, y can be expressed as rational functions of $\hat{x}, \hat{y}$ and, conversely, $\hat{x}, \hat{y}$ as rational functions of x, y . This gives a birational map, which is traditionally called the quadratic Cremona transformation, since it maps straight lines to lines of the second order [10, no. 17].

Example 1. Bilinear equations

$$\hat{x}x = y, \quad \hat{y}y = 1$$

define a birational transformation of the plane that maps the line

$$a\hat{x} + b\hat{y} + c = 0$$

to a second-order line

$$ay^2 + bx + cxy = 0.$$

We will denote the projective coordinates of a point (x, y) as $(u : v : w)$, assuming that

$$x = \frac{u}{w}, \quad y = \frac{v}{w}.$$

When using matrix notation, we will consider the projective coordinates of a point $(u : v : w)$ as a column $\mathbf{u}$. With respect to projective coordinates, the equations (1) defining a quadratic transformation can be written as

$$a(\hat{\mathbf{u}}, \mathbf{u}) = 0, \quad b(\hat{\mathbf{u}}, \mathbf{u}) = 0, \quad (2)$$

where a and b are two bilinear forms. We denote by A and B the matrices of the bilinear forms a and b and write the equations (2) as

$$\hat{\mathbf{u}}^T A \mathbf{u} = 0, \quad \hat{\mathbf{u}}^T B \mathbf{u} = 0. \quad (3)$$

We can say that every quadratic transformation is defined by a pair of matrices. This notation for quadratic transformations was proposed in the 1870s by J. Rosanes. Unfortunately, this work went unnoticed. It is mentioned in thick reviews spanning half a century, but leaving Rosanes's notation without attention.

Example 2. The transformation

$$\hat{x}x = y, \quad \hat{y}y = 1$$

from Example 1 in homogeneous coordinates is written as

$$\hat{u}u - \hat{w}v = 0, \quad \hat{v}v - \hat{w}w = 0$$

and, therefore, in Rosanes notation is described by a pair of matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Not every pair of matrices defines a birational transformation. For this to be true, it is necessary and sufficient that the matrices A and B be linearly independent.

Generally speaking, a birational transformation, unlike a linear transformation, can have singular points, that is, points u that correspond to several points $\hat{u}$, and points $\hat{u}$ that correspond to several points u . Nineteenth-century authors called these points fundamental [10, n. 15, 17]. In Rosanes's time, matrix theory was in its infancy, so the connection between fundamental points and the eigenvalue problem was not noted by him. We can formulate it as follows.

Theorem 1. *The quadratic transformation (3) is not uniquely invertible only at those points u that are eigenvectors of the eigenvalue problem.*

$$\mu Au = \lambda Bu, \quad (4)$$

The quadratic transformation (3) is not uniquely invertible only at those points $\hat{u}$ that are eigenvectors of the eigenvalue problem.

$$\mu A^T \hat{u} = \lambda B^T \hat{u}. \quad (5)$$

Here, by an eigenvalue of problem (4) we mean a pair (μ, λ) in which one of the elements is nonzero. Moreover, it is convenient to consider an eigenvalue as a point $(\mu : \lambda)$ on the projective line P_1 . An eigenvalue $(\mu : \lambda) = (1 : 0)$ exists when the determinant $\det A$ is zero.

The eigenvalues of problems (4) and (5) coincide and are the zeros of the determinant

$$\det(\mu A - \lambda B) \quad (6)$$

Definition 1. *Quadratic transformations generated by matrices such that the determinant (6) is not zero at any point $(\mu : \lambda)$ of the projective line P_1 are called regular.*

In the case of algebraically closed fields, this determinant is necessarily zero somewhere, and therefore regular quadratic transformations do not exist. Let us consider such transformations in the simplest case, over a Galois field of size 2, which we will henceforth denote as $\text{GF}(2)$.

3. Regular quadratic transformations of the Fano plane

The affine plane over the Galois field $\text{GF}(2)$ consists of 4 points, while the projective plane consists of 7 points; it is called the Fano plane [11, n. 1.1, 1.10]. For our calculations, we will use the Sage computer algebra system [12], in which the points of the Fano plane are numbered as shown in Figure 1.

The group of all automorphisms of the Fano plane is the symmetric group S_7 [11, n. 1.12]. A regular quadratic transformation is one-to-one at all points of the Fano plane (Theorem 1), so it permutes the points of the Fano plane and, in this sense, is an element of the group S_7 .

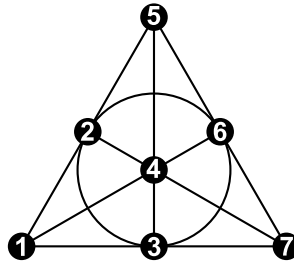


Figure 1. Fano Plane

Example 3. Consider, for example, a transformation that in Rosanes notation is described by a pair of matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

The determinant

$$\det(\mu A - \lambda B) = \mu^3 + \mu^2\lambda + \lambda^3$$

is equal to 1 at all three points of the projective line P_1 , which is traversed by the point $(\mu : \lambda)$. Therefore, this pair of matrices defines a regular quadratic transformation of the Fano plane. The first point has coordinates $(0 : 0 : 1)$, so it corresponds to the point $\hat{u}$, whose coordinates are determined from a system of linear equations

$$\begin{pmatrix} \hat{u} & \hat{v} & \hat{w} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0, \quad \begin{pmatrix} \hat{u} & \hat{v} & \hat{w} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

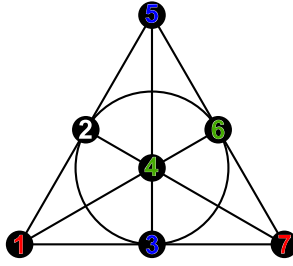
or

$$\hat{w} = 0, \quad \hat{v} = 0.$$

Thus, the first point maps to point $(1 : 0 : 0)$, the 7th in Figure 1. Searching through all seven points of the Fano plane, we find that the quadratic transformation is a permutation

$$(1, 7, 5)(2, 4).$$

Of course, this search was performed in Sage rather than manually. For convenience, we have placed an archive of Sage calculations in [13]. We call the transformation of a pair of matrices defining a quadratic transformation in Rosanes notation into a permutation of the symmetric group S_7 `rosanes_to_permutation(A,B)`.

Figure 2. Permutation $(1, 7)(3, 5)(4, 6)$ Fano plane

4. Analog of Noether's theorem

The composition of two quadratic transformations is a birational transformation, but not necessarily quadratic. Therefore, the set of all quadratic transformations is not a group. The classical Noether theorem states that over $\mathbb{C}$, every birational automorphism of a plane can be represented as a product of quadratic transformations. The key point in the proof of Noether's theorem is the identification of fundamental points of the transformation [14, 15], so it seems that it cannot be extended to the case of finite planes.

We compiled a list of all pairs of matrices for which the determinant (6) cannot be factorized over $\text{GF}(2)$; it contains 8064 pairs. For each such pair, we computed the corresponding permutation (Example 3) and by simple enumeration established the following theorem.

Theorem 2. *Regular quadratic transformations of the Fano plane generate the entire symmetric group S_7 .*

In other words, every automorphism of the Fano plane is a product of permutations that are quadratic transformations. In this sense, the set of all regular permutations is complete; it generates all permutations of points of the Fano plane, which distinguishes it from the set of all possible collinations of the Fano plane that generate a subgroup of S_7 , described in detail in [11, n. 1.12].

5. Classification of regular quadratic transformations as elements of S_7

Since we have enumerated all quadratic transformations in Sage, we can describe the properties of permutations of the symmetric group that are quadratic transformations.

Theorem 3. *Regular quadratic transformations of the Fano plane are permutations of even order.*

To describe regular quadratic transformations as permutations in more detail, recall that the Fano plane has exactly 7 lines, each of which consists of 3 points [11, n. 1.1].

5.1. Second-order permutations

Theorem 4. *A second-order permutation consists of three cycles*

$$(p_1, p_2)(p_3, p_4)(p_5, p_6),$$

where $p_1 \dots p_6$ are the vertices of a complete quadrilateral. Second-order quadratic transformations generate the entire group S_7 .

Example 4. A pair of matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

corresponds to a permutation

$$(1, 7)(3, 5)(4, 6).$$

In Figure 2 three pairs of points are highlighted in color. These points are the vertices of the quadrilateral formed by the four lines 137, 567, 345, and 146.

The square of a second-order permutation is the identity permutation (), which, by Theorem 3, is not a quadratic transformation.

The appearance of the quadrilateral in Theorem 4 is interesting, since this construction is fundamental to the axiomatics of projective geometry [11, n. 1.1].

5.2. Fourth-order permutations

Theorem 1. *Fourth-order transformations are a cycle*

$$(p_1, p_2, p_3, p_4),$$

with three of the four points collinear.

Example 5. The pair of matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

corresponds to the permutation

$$(1, 7, 3, 4),$$

in which the points 1, 7, and 3 lie on the same line. The square of this permutation is

$$(1, 3)(4, 7),$$

it is not among the regular quadratic transformations; its cube is

$$(1, 4, 3, 7),$$

it corresponds to the quadratic transformation

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

5.3. Sixth-order permutations

There are two types of sixth-order permutations. First, there is the cycle

$$(p_1, p_2, p_3, p_4, p_5, p_6),$$

Second,

$$(p_1, p_2, p_3)(p_4, p_5).$$

Not all of them occur; for example, there is no

$$(1, 2, 3, 4, 5, 6).$$

Here, it is significantly more difficult to discern a pattern than in the case of second-order permutations.

6. Constructing a quadratic transformation from a given permutation

Above, in the Example 3, we showed how to determine a permutation from a given quadratic transformation. Let us pose the inverse problem: given a permutation $t \in S_7$, find a quadratic transformation.

So, let a permutation $t \in S_7$ be given. For a pair of matrices A, B defining a quadratic transformation, the following must hold:

$$t(u)^T A u = 0, \quad t(u)^T B u = 0 \quad \forall u \in P_2. \quad (7)$$

Let X be a 3×3 matrix, whose elements are symbolic variables. Then

$$t(u)^T X u = 0 \quad \forall u \in P_2 \quad (8)$$

represent 7 linear homogeneous equations for finding 9 unknowns $x_{11}, \dots, x_{33}$. The solution to this system is a linear subspace in the space of 3×3 matrices. Since its dimension is greater than or equal to $9 - 7 = 2$, there always exist two linearly independent matrices A and B for which (7) holds. This implies the following theorem.

Theorem 2. *There exists a quadratic transformation that permutes the points of the Fano plane in a given way, but this transformation is not always regular.*

From Theorem 3, not every permutation is a regular quadratic transformation. To prevent the formulation of Theorem 2 from appearing to contradict Theorem 3, its formulation specifically stipulates that the resulting quadratic transformations may not be regular.

Example 6. As an example, consider the permutation

$$(1, 2, 3, 4, 5, 6), \quad (9)$$

which does not correspond to any quadratic transformation (see n. 5.3 above). We implemented the method described above as a function `permutation_to_rosanes(t)` in our Sage program [13]. In this case, it yields the expression X as the matrix

$$\begin{pmatrix} 0 & a_4 & a_4 + a_7 + a_8 \\ a_4 + a_8 & a_4 & a_8 \\ a_7 & a_7 & a_8 \end{pmatrix} \quad (10)$$

depending on 3 parameters, i.e., the solution space of the system (8) has dimension 3. However, no two matrices from this space yield a regular quadratic transformation, since the determinant of the expression (10) is equal to

$$a_8 \cdot (a_4 + a_7) \cdot (a_4 + a_7 + a_8).$$

This nonzero polynomial is zero for all possible values of the parameters a_4, a_7, a_8 in the field $\text{GF}(2)$.

According to Theorem 2, every permutation corresponds to some quadratic transformation. Therefore, for some permutations, this transformation inevitably has fundamental points at which the uniqueness of the transformation is violated.

Example 7. Let us return to the permutation (9) from Example 6. Let us take two linearly independent elements in the linear space (10), say,

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (11)$$

and find all possible pairs $(u, \hat{u})$ satisfying the system (3):

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 3 & 3 & 4 & 5 & 6 & 7 & 7 & 7 \\ 2 & 4 & 7 & 3 & 2 & 4 & 7 & 5 & 6 & 1 & 2 & 4 & 7 \end{pmatrix}$$

The first point corresponds not to one point, but to three: points No. 2, 4, and 7, which lie on the same line. Similarly, points No. 3 and 7 correspond not to a single point, but to the same line 247. We can say that uniqueness is violated on the special line 137, to each point of which the degenerate transformation assigns all three points of line 247. If we supplement this permutation with a point selection rule:

$$1 \rightarrow 2, \quad 3 \rightarrow 4, \quad 7 \rightarrow 7, \quad (12)$$

then we obtain the original permutation (9).

When solving the (8) system, we always obtain quadratic transformations that can be predefined at the fundamental points to a one-to-one permutation of the Fano plane. However, the ambiguity here is that choosing a different rule yields a different permutation.

Example 8. For example, the quadratic transformation (11) from Example 7 can be supplemented not only with the rule (12), but also, say, with such a rule

$$1 \rightarrow 2, \quad 3 \rightarrow 7, \quad 7 \rightarrow 4.$$

Then we obtain a permutation

$$(1, 2, 3, 7, 4, 5, 6), \quad (13)$$

different from the original permutation (9).

There is no contradiction with Theorem 2 in obtaining multiple permutations. A new permutation obtained with a different choice of the additional rule will correspond to a different matrix space, but the determinant of all these matrices is zero.

Example 9. The permutation (13) obtained in Example (8) instead of the original permutation (9) corresponds to a degenerate quadratic transformation, the pair of matrices of which can be taken from a linear space.

$$\begin{pmatrix} a_4 + a_7 + a_8 & a_4 & a_4 + a_7 + a_8 \\ & a_4 + a_8 & a_4 & a_8 \\ & & a_7 & a_7 & a_8 \end{pmatrix}.$$

The determinant of this matrix is identically zero.

By choosing a pair of linearly independent matrices in the solution space of the system (8) in various ways, we obtain various quadratic transformations which, after supplementing them with the rules for resolving fundamental points, yield the original permutation.

Example 10. If we choose the basis elements in the three-dimensional linear space (10) differently than in Example 6, we obtain, generally speaking, a different transformation. For example, let us compare with the choice considered in Example 6 a pair of matrices

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and find possible pairs $(u, \hat{u})$ satisfying the system (3):

$$\begin{pmatrix} 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 6 & 7 \\ 2 & 1 & 3 & 7 & 1 & 4 & 6 & 5 & 6 & 7 & 6 & 1 & 7 \end{pmatrix}.$$

Now, uniqueness is violated at points 2, 3, and 4, which are not collinear. Each of these points corresponds to points on different lines:

$$2 \rightarrow 137, \quad 3 \rightarrow 146, \quad 4 \rightarrow 567.$$

We can supplement the permutation with a rule for choosing points on these lines, and as a result, we obtain the original permutation (9).

Thus, given a permutation, it is always possible to find a quadratic transformation, but, firstly, this transformation may not be regular, and, secondly, there may be several suitable transformations.

7. Two classes of irregular quadratic transformations

Solutions to the system (8) yield irregular transformations that have a remarkable property by construction: they can be supplemented with a fundamental point resolution rule, after which they become a simple permutation of points on the Fano plane. However, it is easy to find an example in which such a rule cannot be specified.

Example 11. To contrast with the irregular transformation from Example 6, let us take as an example a pair of matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

The determinant

$$\det(\mu A - \lambda B) = -(\mu^2 - \mu\lambda - \lambda^2)(\mu - \lambda)$$

is not equal to zero, for example, when $(\mu : \lambda) = (1 : 0)$. Let us find all possible pairs $(u, \hat{u})$ satisfying the system (3):

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 3 & 4 & 5 & 6 & 7 \\ 5 & 5 & 2 & 4 & 7 & 3 & 5 & 6 & 1 \end{pmatrix}.$$

Here line 125 contracts to point No. 5, and point No. 3 is broken into line 247. Since three points of line 125 correspond to a single point, No. 5, there is no way to resolve the singularities with an additional rule.

Thus, irregular quadratic transformations are divided into two classes. The first are those whose existence is guaranteed by Theorem 2. These transformations have singular points, but the ambiguity of the correspondence can be restored by imposing additional rules. The second are quadratic transformations that have an irremovable ambiguity. In other words, for an irregular transformation of class 1, there exists a permutation $t \in S_7$ such that (7) holds, while for transformations of class 2, such a permutation does not exist.

The properties of regular transformations and irregular transformations of class 1 can be deduced from the properties of permutations of the group S_7 , which are well studied.

8. Correspondence between the group S_7 and quadratic transformations

There is a certain correspondence between the union of the set of regular transformations and the set of irregular transformations of class 1, on the one hand, and the group S_7 , on the other hand.

Given a quadratic transformation, we can find a permutation. For regular transformations, this is done uniquely (Example 3), while for irregular ones it is ambiguous (Example 10).

Given a given permutation t of S_7 , a quadratic transformation can be found. If the dimension of the solution space of the system (8) of 7 equations with 9 unknowns is 2, then the matrix pencil $\mu A - \lambda B$ is uniquely determined, but not its basis A, B . However, as is easy to see, any basis of this pencil defines a quadratic transformation equivalent to t . Since we consider two transformations that implement identical permutations of points to be equal, different bases of the same matrix pencil define the same quadratic transformation. Therefore, given a permutation, one can uniquely determine the matrix pencil, and from it, the quadratic transformation. Uniqueness is violated when there are linearly dependent equations among the 7 equations of the system (8).

9. Results and discussion

The goal of this study was to use a simple example to understand how strongly Cremona's theory of quadratic transformations over algebraically non-closed fields differs from the classical theory developed over $\mathbb{C}$.

Contrary to our expectations, Noether's theorem remained valid on the Fano plane (Theorem 2), but the quadratic transformations generating the entire automorphism group of the Fano plane were regular quadratic transformations of order 2 (Theorem 4). There is no analogue of such transformations over $\mathbb{C}$.

When integrating dynamic systems using the Kahan method, the degrees of the quadratic Cremona transformations are calculated. A numerical solution is an orbit calculated for a single initial point. If it is calculated over a floating-point implementation of the field $\mathbb{R}$, the result is fast, but with poorly controlled roundoff error. Attempting to calculate the n -th power of the Cremona transform over the field $\mathbb{Q}$ or $\mathbb{Q}(\Delta t)$, and especially with initial data specified in symbolic form, leads to exponential growth in the n -degree of expressions [16–18]. This makes direct methods for finding periodic solutions, which in theory should return periodic solutions or guarantee their absence in a finite number of steps [18], practically unusable.

Theorem 2 allows looking at these problems with some optimism: direct substitution of one birational transformation into another in the case of finite fields leads to a very complex expression, which, however, does not at all exclude the possibility of describing this transformation as an element of the group S_7 and quickly calculating its degree. The point is that Cremona transformations admit various representations, some of which are very complicated. Over algebraically non-closed fields, an additional degree of freedom appears in this situation: one can attempt to reduce the calculation of the degree of a Cremona transformation to the degree of a regular transformation.

Quadratic Cremona transformations are purely algebraic objects, so it seems that their invariants must also be algebraic. The fact that invariant curves of a Cremona transformation may not be algebraic was first noted relatively recently [19]. We encountered this problem when applying Kahan's method to the Volterra–Lotka system [3, 20]. In this case, it is clearly seen that the points lie on some curved line, but it can be proven that this line is not algebraic [20, ex. 5][18, n. 3.3]. There are many different ways to extend the concept of an invariant line to finite planes, especially if we allow fixed points and lines of arbitrarily large order to be added to the invariant sets. However, for the case of the Fano plane, we can say what these sets actually look like.

From Theorem 4 it follows that for quadratic transformations of the second order, an invariant line exists and is a quadrilateral, since all 6 of its points can be described as the roots of an equation obtained by multiplying the left-hand sides of the equations of the 4 straight lines that form its sides. It is interesting to note that all six points of the invariant quadrilateral are movable; the transformation leaves none of them fixed. On the other hand, the fourth-order transformation

$$(1, 7, 3, 4),$$

considered in Example 5 preserves the pair of lines 137 and 146, but one of the five points of this pair, namely point 6, remains fixed.

For fourth-order transformations in Theorem 1, the invariant set is a pair of lines: the line on which three of the four points represented by the transformation lie, and any line passing through the remaining point.

Finally, it is impossible not to notice that the main object in the developed theory turned out to be transformations equivalent to permutations of 7 points of the Fano plane. Irregular transformations of the second class (Example 11) represent a more complex object, not single-valued on the Fano plane. How should it be described in terms of combinatorics?

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О квадратичных преобразованиях плоскости Фано

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Аннотация. Указано значение исследования квадратичных преобразований Кремоны над алгебраически незамкнутыми полями для теории разностных схем Кагана для динамических систем с квадратичной правой частью. Рассмотрены преобразования Кремоны проективной плоскости над полем Галуа размера 2, то есть плоскости Фано. Для описания квадратичных преобразований Кремоны использована нотация, предложенная Я. Розанесом. Описана связь между квадратичными преобразованиями и матричными пучками. Квадратичные преобразования без особых точек названы регулярными. В системе Sage реализованы процедуры, которые переводят квадратичное преобразование в перестановку 7 точек плоскости Фано (элемент симметрической группы S_7) и перестановку 7 точек плоскости в квадратичное преобразование. Перебором всех квадратичных преобразований доказано, что регулярные квадратичные преобразования порождают всю группу перестановок плоскости Фано. Эта теорема является аналогом теоремы Нетера над алгебраически незамкнутым полем. Доказано, что регулярные квадратичные преобразования Кремоны имеют чётный порядок, и дано описание соответствующих им перестановок группы S_7 . Показано, что перестановке всегда отвечает некоторое квадратичное преобразование Кремоны, но не всегда это преобразование является регулярным. Получающиеся при этом квадратичные преобразования можно дополнить правилом, которое разрешает неоднозначность в фундаментальных точках. Приведён пример квадратичного преобразования, для которого такое разрешение сделать нельзя. Отсюда получена естественная классификация квадратичных преобразований плоскости Фано.

Ключевые слова: проективная плоскость, поля Галуа, преобразования Кремоны, метод Кагана