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On a finite-difference scheme defining a birational non-quadratic map between time layers

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Abstract. The article considers reversible difference schemes for dynamical systems based on the system doubling method proposed by V.N. Abrashin and S.N. Sytova. The method duplicates the original variables, leading to an extended system whose finite-difference approximation defines a birational map between time layers. The preservation of algebraic integrals in such schemes is investigated. It is proved that if the original system admits a homogeneous quadratic first integral, the corresponding bilinear form is exactly preserved by the discrete scheme. This property is demonstrated on the Jacobi oscillator, where the geometric mean of the duplicated variables ensures exact conservation of the quadratic integral. A more detailed analysis is performed on the non-trivial Vanhaecke system, an integrable Hamiltonian system with two degrees of freedom and higher-degree polynomial integrals. Numerical experiments carried out in the computer algebra system Sage using the package `fdm.sage` confirm that the two copies oscillate synchronously around the exact values of the first integrals, and averaging reduces the oscillation amplitude. For separable Hamiltonian systems, the scheme is shown to be symplectic. The results obtained allow recommending the doubling method for constructing stable and structure-preserving numerical integrators for a wide class of dynamical systems with polynomial right-hand sides, including high-dimensional systems.

Key words and phrases: dynamical system, finite difference scheme, Kahan's method, integrable system, Vanhaecke system

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1. Introduction

Dynamical systems with quadratic right-hand sides can be approximated by finite-difference schemes. Timestep of these schemes is a birational map [1–5]. Appelroth's quadratization allows any dynamical system with a polynomial right-hand side to be transformed into a system with a quadratic right-hand side to which the Kahan scheme can be applied [6]. Consequently, a very broad class of continuous dynamical systems admits approximation by t -symmetric finite-difference schemes that define a birational map between time layers. Following [5], we refer to such schemes as reversible. This property is of fundamental physical relevance. The time evolution of a dynamical system should, in general, be reversible: every initial state corresponds to exactly one final state, and every final state comes from exactly one initial state. Surprisingly, although the discrete-time methods often preserve one-to-one correspondence, it is not always true for the original continuous systems over the different time intervals [5].

It is worth mentioning that in the papers above the time step was implemented using a quadratic Cremona transformation. However, in the 1980s, V. N. Abrashin proposed a multicomponent alternating-direction method, which was successfully applied by S. N. Sytova. This method was used for modeling the nonlinear dynamics of electromagnetic radiation generated by charged particle beams in multidimensional spatially periodic structures [7, n. 4]. Like Appelroth's quadratization, this approach also introduces extra variables. However, instead of using a complicated substitution, it does so in a very simple and clear way: by duplicating every original variable. Because of this straightforward construction, we call it the system doubling method. A closer look shows that this method can be used almost for any dynamical system to build a finite-difference scheme where each time step is described by a Cremona transformation. In this paper, we give a general description of the doubling method and explain its key algebraic properties.

2. Methods

2.1. System doubling method

Let $\mathbf{x} = (x_1, \dots, x_n)$ and there is a dynamical system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}). \quad (1)$$

For simplicity, assume that the right-hand sides in (1) are polynomials.

Introduce a new set of variables \mathbf{x}' , which duplicate the original set of variables \mathbf{x} , and form the doubled system:

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}'), \\ \frac{d\mathbf{x}'}{dt} = \mathbf{f}(\mathbf{x}). \end{cases} \quad (2)$$

We formulate the equivalence of these systems as a theorem.

Theorem 9. *Every solution $\mathbf{x} = \mathbf{h}(t)$ of system (1) extends to a solution $\mathbf{x} = \mathbf{x}' = \mathbf{h}(t)$ of the doubled system (2). Conversely, any solution of system (2) that satisfies the initial condition $\mathbf{x} = \mathbf{x}'$ at some time t_0 satisfies the equation $\mathbf{x}(t) = \mathbf{x}'(t)$ for all t , and $\mathbf{x} = \mathbf{x}(t)$ is a solution of the original system (1).*

Proof. (i) Suppose $\mathbf{x} = \mathbf{h}(t)$ is a solution of system (1). Then

$$\frac{d\mathbf{h}}{dt} = \mathbf{f}(\mathbf{h}).$$

Substituting $\mathfrak{x}(t) = \mathfrak{x}'(t) = \mathfrak{h}(t)$ into system (2), both equations are satisfied identically, since

$$\frac{d\mathfrak{x}}{dt} = \frac{d\mathfrak{h}}{dt} = \mathfrak{f}(\mathfrak{h}) = \mathfrak{f}(\mathfrak{x}'), \quad \frac{d\mathfrak{x}'}{dt} = \frac{d\mathfrak{h}}{dt} = \mathfrak{f}(\mathfrak{h}) = \mathfrak{f}(\mathfrak{x}).$$

Thus, $(\mathfrak{h}(t), \mathfrak{h}(t))$ is a solution of the doubled system (2).

(ii) Conversely, let $(\mathfrak{x}(t), \mathfrak{x}'(t))$ be a solution of (2) satisfying the initial condition $\mathfrak{x}(0) = \mathfrak{x}'(0)$. Define the differences

$$u_i(t) = x'_i(t) - x_i(t), \quad i = 1, \dots, n,$$

and collect them into the vector $\mathbf{u}(t) = \mathfrak{x}'(t) - \mathfrak{x}(t)$. Differentiating gives

$$\frac{du_i}{dt} = \frac{dx'_i}{dt} - \frac{dx_i}{dt} = f_i(\mathfrak{x}) - f_i(\mathfrak{x}').$$

Since each component f_i is a polynomial (by assumption), the difference $f_i(\mathfrak{x}) - f_i(\mathfrak{x}')$ can be expressed as a linear combination of the differences $x'_j - x_j = u_j$ with polynomial coefficients. More precisely, there exist polynomials $g_{ij}(\mathfrak{x}, \mathfrak{x}')$ such that

$$f_i(\mathfrak{x}) - f_i(\mathfrak{x}') = \sum_{j=1}^n g_{ij}(\mathfrak{x}, \mathfrak{x}') u_j.$$

Hence, $\mathbf{u}(t)$ satisfies the linear homogeneous system

$$\frac{du_i}{dt} = \sum_{j=1}^n g_{ij}(\mathfrak{x}(t), \mathfrak{x}'(t)) u_j, \quad i = 1, \dots, n.$$

This system has continuous (in fact, smooth) coefficients along the trajectory $(\mathfrak{x}(t), \mathfrak{x}'(t))$, and the initial condition $\mathbf{u}(0) = 0$ holds by hypothesis. By uniqueness of solutions to linear ODEs, it follows that $\mathbf{u}(t) \equiv 0$ for all t in the interval of existence. Therefore,

$$\mathfrak{x}(t) = \mathfrak{x}'(t) \quad \text{for all } t.$$

Substituting this identity into the first equation of (2) results in

$$\frac{d\mathfrak{x}}{dt} = \mathfrak{f}(\mathfrak{x}),$$

so $\mathfrak{x}(t)$ is indeed a solution of the original system (1). □

Remark 1. The polynomials u_i arising in the proof are a natural generalization of Darboux polynomials [8–10].

System (2) is more complex than system (1), yet it admits a straightforward finite-difference discretization:

$$\begin{cases} \hat{\mathfrak{x}} - \mathfrak{x} = \mathfrak{f}(\mathfrak{x}') \Delta t, \\ \hat{\mathfrak{x}}' - \mathfrak{x}' = \mathfrak{f}(\hat{\mathfrak{x}}) \Delta t. \end{cases} \quad (3)$$

This scheme defines a birational map between the points $(\mathfrak{x}, \mathfrak{x}')$ and $(\hat{\mathfrak{x}}, \hat{\mathfrak{x}}')$.

In contrast to Kahan's method, the scheme (3) is not t -symmetric [5]. However, it possesses a generalized analogue of t -symmetry: the equations (3) are invariant under the transformation

$$\Delta t \mapsto -\Delta t, \quad \mathfrak{x} \mapsto \hat{\mathfrak{x}}', \quad \mathfrak{x}' \mapsto \hat{\mathfrak{x}}.$$

We shall refer to this property as generalized t -symmetry.

We have implemented the doubling-based scheme (3) in our fdm package for SAGE. All computations presented below were made using this package.

2.2. Quadratic integrals

Theorem 10. *Suppose that system (1) admits a homogeneous quadratic first integral*

$$v = \sum_{i,j} v_{ij} x_i x_j.$$

Then the doubled system (2) possesses the first integral

$$w = \sum_{i,j} v_{ij} x_i x'_j,$$

which is exactly preserved by the discrete scheme (3), i.e.,

$$w(\hat{\mathbf{x}}, \hat{\mathbf{x}}') = w(\mathbf{x}, \mathbf{x}').$$

Proof. We first observe that

$$v = \mathbf{x}^T V \mathbf{x},$$

where V is the symmetric matrix representing the quadratic form v . By assumption,

$$\frac{dv}{dt} = 0,$$

along solutions of (1), i.e.,

$$\dot{\mathbf{f}}(\mathbf{x})^T V \mathbf{x} + \mathbf{x}^T V \dot{\mathbf{f}}(\mathbf{x}) = 0.$$

Since V is symmetric, this simplifies to

$$\mathbf{x}^T V \dot{\mathbf{f}}(\mathbf{x}) = 0. \quad (4)$$

This identity holds for all $\mathbf{x} \in \mathbb{R}_n$, as it is satisfied along every solution of (1).

The quantity w is the bilinear form associated with the quadratic form v , so we can write

$$w = \mathbf{x}^T V \mathbf{x}'.$$

Differentiating w along solutions of the doubled system (2) gives

$$\frac{dw}{dt} = \dot{\mathbf{f}}(\mathbf{x}')^T V \mathbf{x}' + \mathbf{x}^T V \dot{\mathbf{f}}(\mathbf{x}).$$

Each term on the right-hand side vanishes by identity (4). Consequently, w is a first integral of system (2).

We now turn to the finite-difference setting. Consider the increment

$$\hat{w} - w = w(\hat{\mathbf{x}}, \hat{\mathbf{x}}') - w(\mathbf{x}, \mathbf{x}') = \hat{\mathbf{x}}^T V \hat{\mathbf{x}}' - \mathbf{x}^T V \mathbf{x}'.$$

Adding and subtracting the term $\hat{\mathbf{x}}^T V \mathbf{x}'$ gives

$$\hat{w} - w = (\hat{\mathbf{x}}^T V \hat{\mathbf{x}}' - \hat{\mathbf{x}}^T V \mathbf{x}') + (\hat{\mathbf{x}}^T V \mathbf{x}' - \mathbf{x}^T V \mathbf{x}').$$

Using the discrete scheme (3), we have

$$\hat{\mathbf{x}}' - \mathbf{x}' = \dot{\mathbf{f}}(\hat{\mathbf{x}}) \Delta t, \quad \hat{\mathbf{x}} - \mathbf{x} = \dot{\mathbf{f}}(\mathbf{x}') \Delta t,$$

and therefore

$$\hat{\mathbf{x}}^T V \hat{\mathbf{x}}' - \hat{\mathbf{x}}^T V \mathbf{x}' = \hat{\mathbf{x}}^T V (\hat{\mathbf{x}}' - \mathbf{x}') = \Delta t \hat{\mathbf{x}}^T V \dot{\mathbf{f}}(\hat{\mathbf{x}}),$$

$$\hat{\mathbf{x}}^T V \mathbf{x}' - \mathbf{x}^T V \mathbf{x}' = (\hat{\mathbf{x}} - \mathbf{x})^T V \mathbf{x}' = \Delta t \dot{\mathbf{f}}(\mathbf{x}')^T V \mathbf{x}'.$$

By identity (4), both terms vanish. Hence $\hat{w} = w$, which shows that w is exactly preserved by the discrete map (3). \square

2.3. Hamiltonian formulation

Suppose the original system (1) is Hamiltonian, i.e., it can be written as

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad i = 1, \dots, n.$$

Then the doubled system takes the form

$$\begin{cases} \frac{dp_i}{dt} = -\frac{\partial H'}{\partial q'_i}, & \frac{dq_i}{dt} = \frac{\partial H'}{\partial p'_i}, & i = 1, \dots, n, \\ \frac{dp'_i}{dt} = -\frac{\partial H}{\partial q_i}, & \frac{dq'_i}{dt} = \frac{\partial H}{\partial p_i}, & i = 1, \dots, n. \end{cases}$$

where $H = H(p, q)$ and $H' = H(p', q')$. We introduce

$$H_2 = H(p, q) + H(p', q') \quad (5)$$

and rewrite the system in the following way

$$\begin{cases} \frac{dp_i}{dt} = -\frac{\partial H_2}{\partial q'_i}, & \frac{dq'_i}{dt} = \frac{\partial H_2}{\partial p_i}, & i = 1, \dots, n, \\ \frac{dp'_i}{dt} = -\frac{\partial H_2}{\partial q_i}, & \frac{dq_i}{dt} = \frac{\partial H_2}{\partial p'_i}, & i = 1, \dots, n. \end{cases} \quad (6)$$

This combined Hamiltonian allows us to write the doubled system in a compact Hamiltonian form. It is now clear that this system is Hamiltonian, with the same Hamiltonian as in (5). Its symplectic structure is defined by the 1-form

$$\omega = \sum_{i=1}^n (p_i dq'_i + p'_i dq_i).$$

Note the “cross” pairing of coordinates and momenta – each momentum is paired with the other set of coordinates.

The finite-difference scheme obtained by the doubling method reads

$$\begin{cases} \frac{\hat{p}_i - p_i}{\Delta t} = -\frac{\partial H'}{\partial q'_i}(p', q'), & \frac{\hat{q}_i - q_i}{\Delta t} = \frac{\partial H'}{\partial p'_i}(p', q'), & i = 1, \dots, n, \\ \frac{\hat{p}'_i - p'_i}{\Delta t} = -\frac{\partial H}{\partial q_i}(\hat{p}, \hat{q}), & \frac{\hat{q}'_i - q'_i}{\Delta t} = \frac{\partial H}{\partial p_i}(\hat{p}, \hat{q}), & i = 1, \dots, n. \end{cases}$$

Or, using (6), the scheme can be written as

$$\begin{cases} \frac{\hat{p}_i - p_i}{\Delta t} = -\frac{\partial H_2}{\partial q'_i}, & \frac{\hat{q}'_i - q'_i}{\Delta t} = \frac{\partial H_2}{\partial p_i} \Big|_{p=\hat{p}, q=\hat{q}}, & i = 1, \dots, n, \\ \frac{\hat{p}'_i - p'_i}{\Delta t} = -\frac{\partial H_2}{\partial q_i} \Big|_{p=\hat{p}, q=\hat{q}}, & \frac{\hat{q}_i - q_i}{\Delta t} = \frac{\partial H_2}{\partial p'_i}, & i = 1, \dots, n. \end{cases} \quad (7)$$

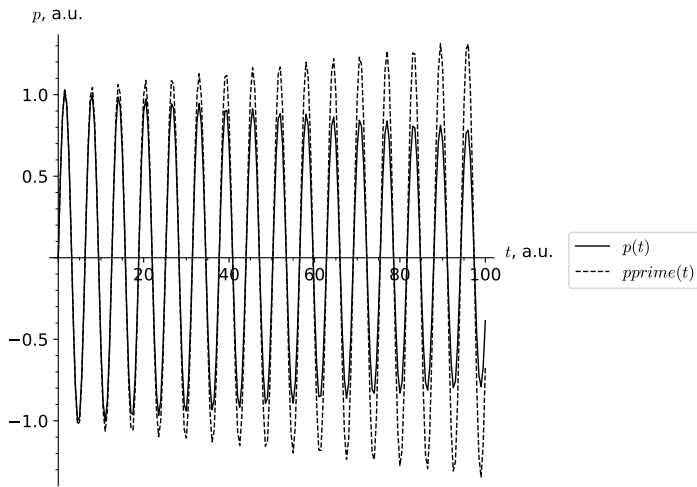


Figure 1. Time evolution of the variables $p(t)$ and $p'(t)$ for the Jacobi oscillator

3. Results

3.1. Jacobi oscillator

To understand the purpose of doubling the variables, consider a simple and well-studied example — the Jacobi oscillator:

$$\begin{cases} \frac{dp}{dt} = qr, \\ \frac{dq}{dt} = -pr, \\ \frac{dr}{dt} = -k^2 pq, \\ p(0) = 0, \\ q(0) = 1, \\ r(0) = 1. \end{cases}$$

It is well known that the solution of this system is periodic and can be expressed in terms of elliptic functions. Figure 1 shows the time dependence of $p(t)$ and $p'(t)$: the amplitude of oscillations of p grows over time, while the amplitude of p' decreases. The exact solution lies somewhere between these two trajectories. The behaviour shown in Fig. 1 closely resembles the classical idea of upper and lower solutions for differential equations, first introduced by Chaplygin [11].

Looking at this figure, one can expect that the average of p and p' approximates the exact solution, while half the absolute difference $\frac{1}{2}|p - p'|$ provides an estimate of the error incurred by the method. The graph of this error estimate is shown in Fig. 2.

Nevertheless, taking arithmetic averages does not produce the behavior typically expected from a geometric integrator. This is particularly clear from the phase portrait in the (p, q) -plane. Figure 3 displays the trajectories of the points (p, q) and (p', q') . As anticipated, (p, q) spirals outward to infinity,

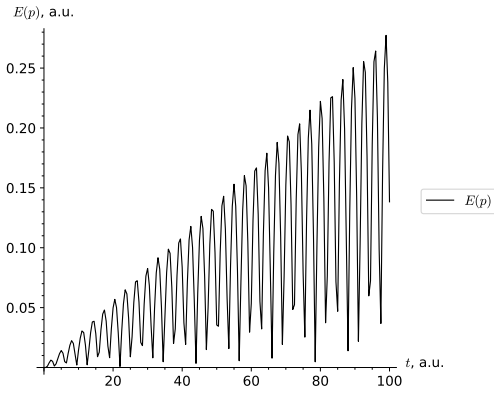


Figure 2. Error function $E(p)$ as a function of time t

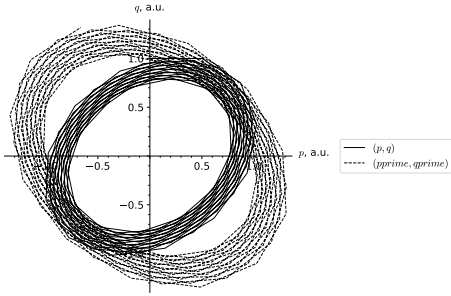


Figure 3. Trajectories of the points (p, q) and (p', q')

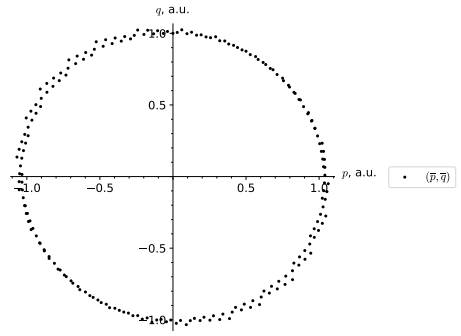


Figure 4. Trajectory of the point (\bar{p}, \bar{q})

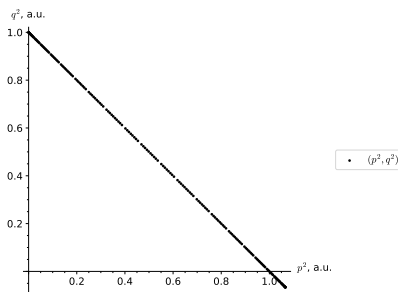


Figure 5. Phase portrait of the Jacobi oscillator in the (q, p) -plane obtained using the geometric mean

whereas (p', q') spirals inward toward the origin. Forming the arithmetic mean — namely, the point

$$(\bar{p}, \bar{q}) = \left(\frac{p + p'}{2}, \frac{q + q'}{2} \right),$$

partially corrects this behavior (see Fig. 4), but the improvement is incomplete. The averaged trajectory still fails to lie on a closed curve, in contrast to the exact periodic orbits or those obtained with Kahan's method. This indicates that the discretization introduces a small amount of numerical dissipation into the system.

However, if we use geometric means in our calculations, the situation improves dramatically. As clearly seen in Fig. 5, the point

$$(\bar{\bar{p}}, \bar{\bar{q}}) = (\sqrt{pp'}, \sqrt{qq'})$$

lies exactly on the circle

$$\bar{\bar{p}}^2 + \bar{\bar{q}}^2 = 1,$$

which is precisely the first integral of the Jacobi oscillator,

$$p^2 + q^2 = 1.$$

Thus, in this case, using geometric averages exactly preserves the original system's motion integral.

As an example of the quadratic integral preservation, the Jacobi oscillator possesses the quadratic first integral

$$v = p^2 + q^2.$$

Consequently, the doubled system admits the invariant

$$w = pp' + qq',$$

which is exactly conserved under the numerical scheme (3). When expressed in terms of geometric means,

$$\bar{\bar{p}} = \sqrt{pp'}, \quad \bar{\bar{q}} = \sqrt{qq'},$$

this invariant takes the form

$$w = \bar{\bar{p}}^2 + \bar{\bar{q}}^2,$$

which explains the exact preservation of the circular phase trajectories observed in Fig. 5.

3.2. Vanhaecke system

To see how integrals of degree higher than two behave under the discrete scheme (3), we look at a well-known example of a completely integrable Hamiltonian system — the Vanhaecke system [12–14], first introduced by P. Vanhaecke in the 1990s. This system has two degrees of freedom and Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2)^2 + \alpha q_1^2 + \beta q_2^2 - \alpha - \beta,$$

where α and β are parameters of the system. This system has two polynomial first integrals:

$$F = (p_2 q_1 - p_1 q_2)^2 - (2q_1^4 + 2q_1^2 q_2^2 + 2\alpha q_1^2 + p_1^2)(\alpha - \beta),$$

and

$$G = (p_2 q_1 - p_1 q_2)^2 + (2q_1^2 q_2^2 + 2q_2^4 + 2\beta q_2^2 + p_2^2)(\alpha - \beta),$$

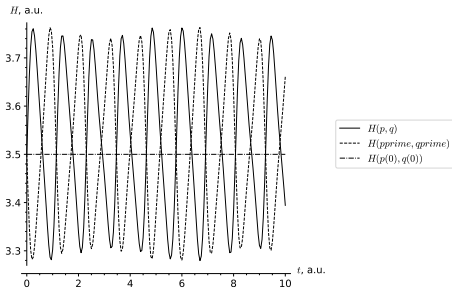


Figure 6. Plots of H and H'

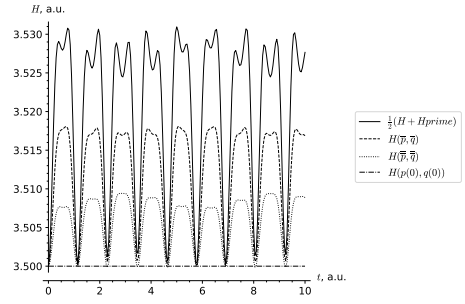


Figure 7. Plot of the averaged Hamiltonian

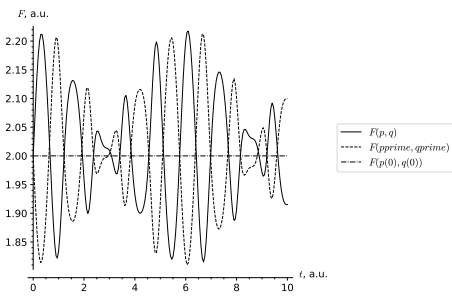


Figure 8. Plots of F and F'

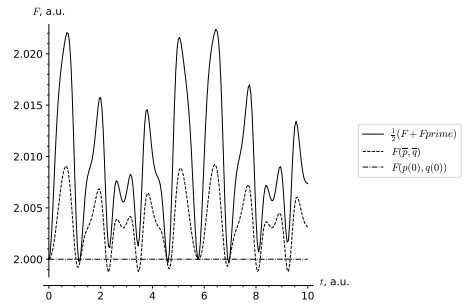


Figure 9. Plots of averaged values of F

which are in involution. Moreover, their difference is proportional to the Hamiltonian H .

We integrated the system using the scheme (3) with the initial conditions

$$\begin{cases} q_1(0) = 0, \\ q_2(0) = 1, \\ p_1(0) = 1, \\ p_2(0) = 0, \end{cases}$$

parameters $\alpha = 1$, $\beta = 2$, and time step $\Delta t = 0.05$.

Figure 6 shows the values of $H = H(p, q)$ and $H' = H(p', q')$. Even with such a small time step, the energy changes noticeably – already in the first decimal place. However, H and H' oscillate out of phase around a value close to the exact one, and – most importantly – neither shows any long-term drift upward or downward.

Averaging in different ways reduces the size of these oscillations. We tried three approaches:

- the simple average $\frac{1}{2}(H + H')$ – shown as the solid line in Fig. 7,
- $H(\bar{p}, \bar{q})$, where $\bar{p} = \frac{p+p'}{2}$ and $\bar{q} = \frac{q+q'}{2}$ – dashed line,
- $H(\bar{\bar{p}}, \bar{\bar{q}})$, where $\bar{\bar{p}} = \sqrt{pp'}$ and $\bar{\bar{q}} = \sqrt{qq'}$ – dotted line.

This keeps the energy with an error of order 10^{-3} , and, most importantly, the energy shows no long-term drift upward or downward.

Figure 8 shows how the integral F behaves over time. When computing the averages (Fig. 9), we used only the first two of the three methods described above. The geometric mean was not used here because the expression for F involves more than just squares of coordinates and momenta, which caused technical difficulties.

The same pattern is clearly visible as in the energy plot: the two copies F and F' oscillate out of phase around the true value, and averaging reduces the oscillation amplitude without introducing any drift.

3.3. Symplecticity

The scheme (7) is a partitioned (or split) finite-difference method [15, 16]. The first $2n$ variables (p, q') and the last $2n$ variables (p', q) are updated using different formulas, which is typical for such methods. Because of this structure, the scheme is expected to be symplectic.

Theorem 11. *If the Hamiltonian splits as*

$$H = T(p) + U(q),$$

then the discrete scheme (7) is symplectic, i.e.,

$$\hat{\omega} - \omega = dS$$

for some function S .

Proof. Let us denote the partial derivatives of T and U by

$$T_i = \frac{\partial T}{\partial p_i}, \quad U_i = \frac{\partial U}{\partial q_i}.$$

Recall that the discrete symplectic form is

$$\hat{\omega} = \sum_{i=1}^n (\hat{p}_i d\hat{q}'_i + \hat{p}'_i d\hat{q}_i).$$

We treat the two parts separately. First, using the update rule from (7),

$$\hat{q}'_i = q'_i + \Delta t T_i(\hat{p}),$$

so

$$\sum \hat{p}_i d\hat{q}'_i = \sum \hat{p}_i d(q'_i + \Delta t T_i(\hat{p})) = \sum \hat{p}_i dq'_i + \Delta t \sum \hat{p}_i dT_i(\hat{p}).$$

The second term is an exact differential. Indeed,

$$\sum \hat{p}_i dT_i(\hat{p}) = d\left(\sum \hat{p}_i T_i(\hat{p}) - T(\hat{p})\right),$$

so we can write

$$\sum \hat{p}_i d\hat{q}'_i = \sum \hat{p}_i dq'_i + dS_1(\hat{p}),$$

where $S_1(\hat{p}) = \Delta t(\sum \hat{p}_i T_i(\hat{p}) - T(\hat{p}))$.

Now use the other half of the scheme: $\hat{p}_i = p_i - \Delta t U_i(q')$, so

$$\sum \hat{p}_i dq'_i = \sum p_i dq'_i - \Delta t \sum U_i(q') dq'_i = \sum p_i dq'_i - dU(q').$$

Putting everything together,

$$\sum \hat{p}_i d\hat{q}_i = \sum p_i dq_i - dU(q') + dS_1(\hat{p}).$$

A similar calculation for the second part gives

$$\sum \hat{p}'_i d\hat{q}_i = \sum p'_i dq_i + dU(\hat{q}) = \sum p'_i dq_i + dS_2(p') + dU(\hat{q}),$$

where $S_2(p') = \Delta t(\sum p'_i T_i(p') - T(p'))$.

Adding both contributions and subtracting the original form

$$\omega = \sum (p_i dq_i + p'_i dq_i),$$

we find that all non-exact terms cancel each other out, and

$$\hat{\omega} - \omega = dS$$

with $S = S_1(\hat{p}) + S_2(p') - U(q') + U(\hat{q})$. Hence the scheme is symplectic. \square

4. Discussion

In general, it is highly convenient when a numerical method inherently includes a built-in error estimator. In contrast, methods like Runge–Kutta require such error estimation techniques to be added separately [17].

Nevertheless, taking arithmetic averages does not produce the behavior typically expected from a geometric integrator. This is particularly clear from the phase portrait in the (p, q) -plane. Figure 3 displays the trajectories of the points (p, q) and (p', q') . As anticipated, (p, q) spirals outward to infinity, whereas (p', q') spirals inward toward the origin.

Forming the arithmetic mean – namely, the point

$$(\bar{p}, \bar{q}) = \left(\frac{p + p'}{2}, \frac{q + q'}{2} \right),$$

partially corrects this behavior (see Fig. 4), but the improvement is incomplete. The averaged trajectory still fails to lie on a closed curve, in contrast to the exact periodic orbits or those obtained with Kahan's method. This indicates that the discretization introduces a small amount of numerical dissipation into the system.

However, if we use geometric means in our calculations, the situation improves dramatically. As clearly seen in Fig. 5, the point

$$(\bar{\bar{p}}, \bar{\bar{q}}) = (\sqrt{pp'}, \sqrt{qq'})$$

lies exactly on the circle

$$\frac{\bar{\bar{p}}^2}{p} + \frac{\bar{\bar{q}}^2}{q} = 1,$$

which is precisely the first integral of the Jacobi oscillator,

$$p^2 + q^2 = 1.$$

Thus, in this case, using geometric averages exactly preserves the original system's motion integral.

In this sense, scheme (3) actually works better than Kahan's method. Unlike the doubling scheme, Kahan's method does not preserve the original integral $p^2 + q^2$. Instead, it conserves a modified expression dependent on Δt . Finding this expression for the Kahan scheme was the main result of the work [18], in [5] this expression was found according to the Lagutinsky method.

Exact preservation is important in many cases. For example, one conservation law for the spinning top simply says that the sum of the squares of the direction cosines is always 1 [19]. Kahan's method preserves a modified expression [20], i.e., adds tiny corrections that don't have a clear geometric meaning. Because scheme (3) preserves this integral exactly, it looks very promising for building geometric integrators — methods that respect the underlying geometry of the problem.

Overall, we can say that the scheme (3) keeps all integrals of the system close to their exact values allowing only small oscillations around them. This is a very good property and suggests that the scheme may be reliable for long-time simulations. At the same time, it is important to note that the energy integral is not preserved exactly, no matter how we average it.

It should also be mentioned that the Vanhaecke system has cubic right-hand sides, so it cannot be directly discretized by Kahan's method. Even if it could, there are no general results guaranteeing that Kahan's method preserves arbitrary polynomial integrals. The only case that is well understood is the preservation of the energy (Hamiltonian) itself [2].

Energy conservation in symplectic integrators has been studied extensively. It is well known that the original Hamiltonian is not preserved exactly. However, one can construct a modified Hamiltonian $H^{(m)}$ that is conserved up to any desired order of accuracy [15, 21]. Because of this, numerical experiments typically show the energy oscillating around a constant value that is close to — but not exactly equal to — the true energy. This is precisely what we observed in the Vanhaecke system. Looking at Fig. 7, one might even guess that the different averaging strategies we used correspond to modified Hamiltonians of different orders.

5. Conclusion

In this work, we studied reversible finite-difference schemes for dynamical systems based on the doubling method introduced by V. N. Abrashin and S. N. Sytova. We carried out a detailed analysis of algebraic integral preservation for two benchmark systems: the classical Jacobi oscillator and the Vanhaecke system.

First of all, quadratic integrals are proven to preserve exactly (Th. 9). In particular, the quadratic integrals for the Jacobi oscillator are preserved exactly if we replace the squares of the quantities with the geometric mean. For comparison's sake, Kahan's method does not preserve the original quadratic integral $p^2 + q^2$, but conserves a modified expression dependent on Δt . Exact preservation is important for geometrical interpretation of expressions like the sum of the squares of the direction cosines.

For Hamiltonian systems, this method is symplectic in the sense of the theorem 11. Thus the original Hamiltonian is not preserved exactly, but one can construct a modified Hamiltonian $H^{(m)}$ that is conserved up to any desired order. It seems that the different averaging strategies we used correspond to modified Hamiltonians of different orders. In any way, the scheme keeps all integrals of the Vanhaecke system close to their exact values allowing only small oscillations around them. Thus the scheme may be reliable for long-time simulations.

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Об одной разностной схеме, задающей бирациональное, но не квадратичное соответствие между слоями

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Аннотация. В статье рассматриваются обратимые разностные схемы для динамических систем, основанные на методе удвоения системы, предложенном В. Н. Абрашиным и С. Н. Сытовой. Метод заключается в дублировании исходного набора переменных, что позволяет перейти к расширенной системе, для которой строится конечно-разностная аппроксимация, задающая бирациональное отображение между соседними временными слоями. Исследуется сохранение алгебраических интегралов таких схем. Доказывается, что если исходная система допускает однородный квадратичный первый интеграл, то соответствующая билинейная форма является точным интегралом дискретной схемы. Это свойство демонстрируется на классическом примере осциллятора Якоби, где схема сохраняет точную величину, выраженную через среднее геометрическое дублированных переменных, воспроизводя корректную геометрию фазовых траекторий. Более глубокий анализ проводится на примере нетривиальной системы Ванхаеке — интегрируемой гамильтоновой системы с двумя степенями свободы, обладающей полиномиальными интегралами высших степеней, интегрируемость которой выражается через абелевы функции. Численные эксперименты, реализованные в системе компьютерной алгебры Sage с использованием специализированного пакета `fdm.sage`, подтверждают, что при дискретизации методом удвоения две копии системы синхронно колеблются около точных значений первых интегралов, а применение усреднения снижает амплитуду колебаний. Для сепарабельных гамильтоновых систем показана симплектичность схемы. Полученные результаты позволяют рекомендовать метод удвоения для построения устойчивых и структуросохраняющих численных интеграторов для широкого класса динамических систем с полиномиальными правыми частями, включая системы высокой размерности.

Ключевые слова: динамические системы, конечные разности, схема Кагана, интегрируемые системы, система Ванхаеке