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Dual quaternion representation of geometrical motion in 3D space

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Abstract. *Background* In a previous article we discussed the use of dual quaternions for modeling points, lines and planes and solving standard geometric problems. This article is a logical continuation and reveals the use of dual quaternions to describe isometries of three-dimensional space. *Purpose* The derivation of all necessary formulas for the screw motion of points, straight lines and planes, as well as reflection relative to the plane. Refinement of notation and formalism. *Method* The algebra of dual numbers, quaternions and dual quaternions is used, as well as elements of the theory of screws and sliding vectors. *Results* Formulas for rotation, translation, reflection, helical motion, and mirror rotation are obtained and systematized. *Conclusions* Dual quaternions can serve as a full-fledged tool for describing helical motion in space. Due to the possibility of expressing dual quaternion operations in terms of standard vector and scalar products, the formulas obtained allow for effective software implementation.

Key words and phrases: natural modeling, reproducible research, research as code

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1. Introduction

In the previous article [1], we considered parabolic biquaternions (dual quaternions), the discovery of which is attributed to W. Clifford and systematic study began later in the works of E. Study [2, 3] and A. P. Kotelnikov [4] including under the guise of the theory of screws [5–9].

This article logically continues the previous one [1] and focuses on the application of dual quaternions to the description of isometries (proper and improper motions) of three-dimensional space. There are two main objectives.

- Output the necessary formulas for calculating all possible movements of three-dimensional space for points, lines and planes. These movements include rotations, translations (parallel transfers), and mirror symmetries.
- In the process of deducing formulas, illustrate the work of mathematical formalism, for which the conclusion is given in great detail, with all the details. A number of quaternion formulas have also been preliminarily obtained, also for the purpose of illustrating the notation.

As a novelty of the paper we can mention the description of not only the movement of lines (or screws, as it is done in the fundamental monograph [10]), but also points and planes, as well as consideration of reflections, which is rare in the literature. In the process, we relied on books [11–13], however, we also provide a number of new formulas. We also base our conclusion on the principle of Kotelnikov–Study transference, which is apparently unknown in foreign papers.

Dual quaternions have significant applied importance, as evidenced by publications on the topic [14–19]. In this article, we do not provide any software implementations, since adding this material would make it necessary to shorten the calculations. Examples of software implementation and an illustration of the operation of all the formulas obtained are planned in further publications by the authors.

1.1. Structure of the paper

The article consists of an introduction, 2 parts and a conclusion listing the results.

In the first part of the article, the quaternionic rotation formula is derived. The conclusion is based on the Rodrigues formula, which can be obtained from relatively elementary geometric constructions. The relation of quaternions to rotation matrices is shown below. Much attention is paid to the algorithm for calculating the rotation quaternion according to a given matrix. This algorithm takes into account the problem of rounding when working with floating-point numbers. In the last paragraph, a formula is derived for reflection relative to a plane, including one that does not pass through the origin.

The second part is the main one. It begins with the formulation of the Kotelnikov–Rudy transference principle. Next, this principle is applied to the quaternionic formula of rotation, which allows you to immediately obtain a dual quaternionic formula of helical motion for a straight line. The dual quaternion of helical motion is written out explicitly and a number of properties are proved for it. Next, it is divided into translational and rotational parts, and their actions on the direct line are studied separately. By creating compositions from these parts, it is possible to describe more complex cases, for example, the mismatch of the axis of rotation with the axis of translation.

Further, using the duality principle allows us to generalize the formula of helical motion to the cases of points and planes. For these cases, explicit formulas for pure translation and pure rotation are also given in detail.

This part ends with the derivation of dual quaternion formulas for reflecting straight lines, points, and planes relative to an arbitrary plane.

1.2. Notations and conventions

The following naming conventions are accepted in this article

- Quaternions are indicated by lowercase Latin letters from the end of the alphabet: p, q, r. The components of the quaternions are indicated by the same letters, but with the indexes p_0, p_1, etc.
- Dual quaternions are indicated by uppercase Latin letters from the end of the alphabet: P, Q, R. The components of the quaternions are indicated by the same letters, but with the indexes P_0, P_1, etc.
- Vectors and pure quaternions are indicated by lowercase bold Latin letters: $\mathbf{q}$, $\mathbf{v}$, etc.
- Pure dual quaternions are indicated by uppercase bold Latin letters: $\mathbf{Q}$, $\mathbf{V}$, etc.
- Individual scalars (real numbers) are denoted by the Greek letters α, β, etc.

To avoid ambiguity in the notation system, we do not use multiple quaternions and dual quaternions designated by the same letter, but distinguished by an index. The only exceptions are dual quaternions of points, lines, and planes, the components of which are denoted by letters other than the letters denoting these dual quaternions.

1.3. Description of geometric motion in three-dimensional space using quaternions

By motion in Euclidean space we mean an affine transformation that preserves the scalar product (metric). An affine transformation can be written as:

Motion in three-dimensional space is reduced to three:

- translation (parallel translation);
- rotation;
- reflection.

The linear part of the affine transformation is responsible for the rotation.

1.4. Rotations using quaternions

1.4.1. Sandwich formula for quaternion rotation of a point

The quaternion formula for rotating a point about an axis passing through the origin is widely known [20, 21]. Let a unit quaternion be given

$$\lambda = \lambda_0 o + \ = \lambda_0 o + \lambda_1 \mathrm{i} + \lambda_2 \mathrm{j} + \lambda_3 \mathrm{k}, \quad \lambda_0 = \cos\frac{\theta}{2}, \quad \lambda_i = \sin\frac{\theta}{2} a_i,$$

where $\mathbf{a} = (a_1, a_2, a_3)^T$ is the direction vector of the rotation axis passing through the origin, and θ is the magnitude of the rotation angle. Then, for a point P with homogeneous coordinates $\vec{\mathrm{p}} = (x, y, z \mid w)$, given in quaternion form as $p = wo + x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$, the following sandwich formula holds:

$$p' = \lambda p \lambda^*,$$

where p' is a quaternion that specifies the homogeneous coordinates of the point P', into which the original point P has passed after the rotation.

Note that the interpretation of the scalar component of the quaternion p as a homogeneous coordinate w is not often encountered in the literature, and here we rely on the sources [12].

Let us prove the above formula purely algebraically using the Rodrigues formula in the Rodrigues–Hamilton form. Recall that this formula in the space $\mathbb{R}^3$ is written as

$$\mathbf{p}' = \mathbf{p} + 2\lambda_0 \times \mathbf{p} + 2 \times (\times \mathbf{p}),$$

where $\mathbf{p}$ is the radius vector defining the initial position of the point P, and λ_0, λ_1, λ_2, λ_3 are the Rodrigues–Hamilton parameters that completely coincide with the components of the quaternion λ and obey the condition $\lambda_0^2 + \|\|^2 = 1$. The proof of the Rodrigues–Hamilton formula does not use the concept of quaternions in any way, so if we transform it so as to replace vector multiplication with quaternion multiplication and the components λ_i with the quaternion λ, then we will also prove the quaternion sandwich formula.

We transform the Rodrigues–Hamilton formula using the normalization $\lambda_0^2 + \|\|^2 = 1$

$$\mathbf{p}' = (\lambda_0^2 + \|\|^2)\mathbf{p} + 2\lambda_0 \times \mathbf{p} + 2 \times (\times \mathbf{p}) = \lambda_0^2\mathbf{p} + \|\|^2\mathbf{p} + 2\lambda_0 \times \mathbf{p} + 2 \times (\times \mathbf{p}).$$

Using the formula $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a},\mathbf{c}) - \mathbf{c}(\mathbf{a},\mathbf{b})$ we write

$$\times(\times\mathbf{p}) = (,\mathbf{p}) - \mathbf{p}(,) = (,\mathbf{p}) - \mathbf{p}\|\|^2,$$

and substitute into the Rodrigues–Hamilton formula

$$\mathbf{p}' = \lambda_0^2\mathbf{p} + \|\|^2\mathbf{p} + 2\lambda_0\times\mathbf{p} + (,\mathbf{p}) - \mathbf{p}\|\|^2 + \times(\times\mathbf{p}) = \lambda_0^2\mathbf{p} + 2\lambda_0\times\mathbf{p} + (,\mathbf{p}) + \times(\times\mathbf{p}).$$

We make the following transformation: $\times(\times\mathbf{p}) = -(\times\mathbf{p})\times$ and add a dummy term $(\times\mathbf{p},) = 0$. Equality to zero is true due to $\times\mathbf{p}\perp$, then

$$\mathbf{p}' = \lambda_0^2\mathbf{p} + 2\lambda_0\times\mathbf{p} + (,\mathbf{p}) - \big(-(\times\mathbf{p},) + (\times\mathbf{p})\times\big).$$

Let us now temporarily, within the limits of this derivation, denote quaternion multiplication by the symbol $\circ$ and interpret all vectors in the formula as pure quaternions. We write

$$-(\times\mathbf{p},) + (\times\mathbf{p})\times = (\times\mathbf{p})\circ \quad \text{и} \quad (,\mathbf{p}) = (,\mathbf{p})\circ.$$

and also replace $2\times\mathbf{p} = \circ\mathbf{p} - \mathbf{p}\circ$ and continue transforming the Rodrigues–Hamilton formula:

$$\begin{aligned}\mathbf{p}' &= \lambda_0^2\mathbf{p} + 2\lambda_0\times\mathbf{p} + (,\mathbf{p})\circ + (\times\mathbf{p})\circ = \lambda_0^2\mathbf{p} + \lambda_0(\circ\mathbf{p} - \mathbf{p}\circ) - \big(-(,\mathbf{p}) + \times\mathbf{p}\big)\circ = \\ &= \lambda_0^2\mathbf{p} + \lambda_0(\circ\mathbf{p} - \mathbf{p}\circ) - (\circ\mathbf{p})\circ = \lambda_0^2\mathbf{p} - \lambda_0\mathbf{p}\circ + \lambda_0\circ\mathbf{p} - (\circ\mathbf{p})\circ = \lambda_0(\lambda_0\mathbf{p} - \mathbf{p}\circ) + \lambda_0\circ\mathbf{p} - (\circ\mathbf{p})\circ = \\ &= \lambda_0\mathbf{p}_0(\lambda_0 -) + \circ\mathbf{p}\circ(\lambda_0 -) = (\lambda_0\mathbf{p} + \circ\mathbf{p})\circ(\lambda_0 -) = (\lambda_0 +)\circ\mathbf{p}\circ(\lambda_0 -)\end{aligned}$$

As a result, we obtained a sandwich formula:

$$\mathbf{p}' = (\lambda_0 +)\circ\mathbf{p}\circ(\lambda_0 -) = \lambda\circ\mathbf{p}\circ\lambda^*.$$

Due to $\lambda\lambda^* = 1$ we can finally write:

$$p' = w + \mathbf{p}' = w\lambda\lambda^* + \lambda\circ\mathbf{p}\circ\lambda^* = \lambda\circ(w + \mathbf{p})\circ\lambda^* \Rightarrow p' = \lambda\circ p\circ\lambda^*.$$

1.4.2. Calculating a rotation matrix given a rotation quaternion

Let's use the Rodrigues formula, written in terms of the Rodrigues–Hamilton coefficients

$$\mathbf{p}' = \mathbf{p} + 2\lambda_0\times\mathbf{p} + 2\times\times\mathbf{p},$$

but let us represent the vector product in matrix form as follows:

$$\times\mathbf{p} = \Lambda\mathbf{p} = \begin{bmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

$$\times\times\mathbf{p} = \Lambda(\Lambda\mathbf{p}) = \Lambda^2\mathbf{p} = \begin{bmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} =$$

$$= \begin{bmatrix} -(\lambda_2^2+\lambda_3^2) & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_1\lambda_2 & -(\lambda_1^2+\lambda_3^2) & \lambda_2\lambda_3 \\ \lambda_1\lambda_3 & \lambda_2\lambda_3 & -(\lambda_1^2+\lambda_2^2) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix},$$

where Λ is a matrix composed of the components of the vector part of the unit quaternion λ. Now Rodrigues' formula can be rewritten in matrix form as follows:

$$\mathbf{p}' = I\mathbf{p} + 2\lambda_0\Lambda\mathbf{p} + 2\Lambda^2\mathbf{p} = (I + 2\lambda_0\Lambda + 2\Lambda^2)\mathbf{p},$$

$$I + 2\lambda_0\Lambda + 2\Lambda^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -2\lambda_0\lambda_3 & 2\lambda_0\lambda_2 \\ 2\lambda_0\lambda_3 & 0 & -2\lambda_0\lambda_1 \\ -2\lambda_0\lambda_2 & 2\lambda_0\lambda_1 & 0 \end{bmatrix} +$$

$$= \begin{bmatrix} -2(\lambda_2^2+\lambda_3^2) & 2\lambda_1\lambda_2 & 2\lambda_1\lambda_3 \\ 2\lambda_1\lambda_2 & -2(\lambda_1^2+\lambda_3^2) & 2\lambda_2\lambda_3 \\ 2\lambda_1\lambda_3 & 2\lambda_2\lambda_3 & -2(\lambda_1^2+\lambda_2^2) \end{bmatrix} = \begin{bmatrix} 1-2(\lambda_2^2+\lambda_3^2) & 2(\lambda_1\lambda_2-\lambda_0\lambda_3) & 2(\lambda_0\lambda_2+\lambda_1\lambda_3) \\ 2(\lambda_0\lambda_3+\lambda_1\lambda_2) & 1-2(\lambda_1^2+\lambda_3^2) & 2(\lambda_2\lambda_3-\lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3-\lambda_0\lambda_2) & 2(\lambda_0\lambda_1+\lambda_2\lambda_3) & 1-2(\lambda_1^2+\lambda_2^2) \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2 & 2(\lambda_1\lambda_2-\lambda_0\lambda_3) & 2(\lambda_0\lambda_2+\lambda_1\lambda_3) \\ 2(\lambda_0\lambda_3+\lambda_1\lambda_2) & \lambda_0^2-\lambda_1^2+\lambda_2^2-\lambda_3^2 & 2(\lambda_2\lambda_3-\lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3-\lambda_0\lambda_2) & 2(\lambda_0\lambda_1+\lambda_2\lambda_3) & \lambda_0^2-\lambda_1^2-\lambda_2^2+\lambda_3^2 \end{bmatrix}.$$

The last matrix was obtained by replacing $1 = \lambda_0^2 + \|\mathbf{q}\| = \lambda_0^2 + \lambda_1^2 + \lambda_2^2\lambda_3^2$.

1.4.3. Calculating a rotation quaternion from a given rotation matrix

Here we present an algorithm for calculating the coefficients of a quaternion λ given a rotation matrix R. The algorithm follows the calculation method described in [22, pp. 90–94], which allows for some compensation for the loss of precision when working with floating-point numbers.

Let us write the rotation matrix in quaternion form

$$R = \begin{bmatrix} 1-2(\lambda_2^2+\lambda_3^2) & 2(\lambda_1\lambda_2-\lambda_0\lambda_3) & 2(\lambda_1\lambda_3-\lambda_0\lambda_2) \\ 2(\lambda_0\lambda_3+\lambda_1\lambda_2) & 1-2(\lambda_1^2+\lambda_3^2) & 2(\lambda_2\lambda_3-\lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3-\lambda_0\lambda_2) & 2(\lambda_2\lambda_3+\lambda_0\lambda_1) & 1-2(\lambda_1^2+\lambda_2^2) \end{bmatrix}$$

where $\lambda = \cos\dfrac{\theta}{2} + \sin\dfrac{\theta}{2},\ = \lambda_1\mathrm{i} + \lambda_2\mathrm{j} + \lambda_3\mathrm{k}, \lambda_0 = \cos\dfrac{\theta}{2}$ и $|\lambda| = 1$.

Note that λ and $-\lambda$ define the same rotation, since the minus sign is neutralized in the sandwich formula

$$(-\lambda)p(-\lambda)^* = \lambda p\lambda^*.$$

This property allows us to choose the sign before the quaternion so that the scalar part is always positive. If $\lambda_0 > 0$, then we can store only three vector components $\lambda_1, \lambda_2, \lambda_3$, and calculate the scalar part from the unity condition $\lambda_0 = \sqrt{1 - \|\|} = \sqrt{1 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2}$ and choose a positive sign before the root.

Let us represent the matrix R as coefficients r^i_j, where $i = 1, 2, 3$ is the column index, and j is the row index

$$R = \begin{bmatrix} r^1_1 & r^1_2 & r^1_3 \\ r^2_1 & r^2_2 & r^2_3 \\ r^3_1 & r^3_2 & r^3_3 \end{bmatrix}$$

Let us show that the scalar element λ_0 can be calculated through the trace of the matrix.

$$r^1_1 + r^2_2 + r^3_3 = 3 - 2(\lambda_2^2 + \lambda_3^2 + \lambda_1^2 + \lambda_3^2 + \lambda_1^2 + \lambda_2^2) = 3 - 4(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = 3 - 4\|\|^2 = 3 - 4(1 - \lambda_0^2) = 4\lambda_0^2 - 1.$$

Next we write $4\lambda_0^2 = 1 + r^1_1 + r^2_2 + r^3_3 = 1 + \operatorname{Tr} R$, which will finally give an expression for λ_0

$$\lambda_0 = \pm\frac{1}{2}\sqrt{1 + \operatorname{Tr} R}.$$

Note that the expression under the root is always positive, since $1 + \operatorname{Tr} R = 4\lambda_0^2 \geqslant 0$.

To calculate λ_1, λ_2, λ_3 we write

$$\begin{aligned} r^3_2 - r^2_3 &= 2(\lambda_2\lambda_3 + \lambda_0\lambda_1 - \lambda_2\lambda_3 + \lambda_0\lambda_1) = 4\lambda_0\lambda_1, \\ r^1_3 - r^3_1 &= 2(\lambda_1\lambda_3 + \lambda_0\lambda_2 - \lambda_1\lambda_3 + \lambda_0\lambda_2) = 4\lambda_0\lambda_2, \\ r^2_1 - r^1_2 &= 2(\lambda_0\lambda_3 + \lambda_1\lambda_2 - \lambda_1\lambda_2 + \lambda_0\lambda_3) = 4\lambda_0\lambda_3. \end{aligned}$$

As a result, we obtain a set of formulas

$$\begin{aligned} 4\lambda_0\lambda_1 &= r^3_2 - r^2_3, \\ 4\lambda_0\lambda_2 &= r^1_3 - r^3_1, \qquad \Longleftrightarrow \qquad 4\lambda_0\lambda_i = r^{i-1}_{i+1} - r^{i+1}_{i-1}, \quad i = 1, 2, 3. \\ 4\lambda_0\lambda_3 &= r^2_1 - r^1_2, \end{aligned} \tag{1}$$

If $|\lambda_0|$ is large enough, say $|\lambda_0| > 1/2$, then the quaternion coefficients can be calculated using the following formulas:

$$\lambda_0 = \pm\sqrt{1 + \operatorname{Tr} R}, \quad \lambda_1 = \frac{r^3_2 - r^2_3}{4\lambda_0}, \quad \lambda_2 = \frac{r^1_3 - r^3_1}{4\lambda_0}, \quad \lambda_3 = \frac{r^2_1 - r^1_2}{4\lambda_0}.$$

If $|\lambda_0|$ is small, say $|\lambda_0| \leqslant 1/2$, then a more sophisticated calculation scheme will have to be used.

We will need a set of three groups of formulas. To obtain the first group, we write

$$\begin{aligned} r^1_1 - r^2_2 - r^3_3 + 1 &= 1 - 2\lambda_2^2 - 2\lambda_3^2 - 1 + 2\lambda_1^2 + 2\lambda_3^2 - 1 + 2\lambda_1^2 + 2\lambda_2^2 + 1 = 4\lambda_1^2, \\ -r^1_1 + r^2_2 - r^3_3 + 1 &= -1 + 2\lambda_2^2 + 2\lambda_3^2 + 1 - 2\lambda_1^2 - 2\lambda_3^2 - 1 + 2\lambda_1^2 + \lambda_2^2 + 1 = 4\lambda_2^2, \\ -r^1_1 - r^2_2 + r^3_3 + 1 &= -1 + 2\lambda_2^2 + 2\lambda_3^2 - 1 + 2\lambda_1^2 + 2\lambda_3^2 + 1 - 2\lambda_1^2 - 2\lambda_2^2 + 1 = 4\lambda_3^2, \end{aligned}$$

obtained the following set of formulas:

$$4\lambda_1^2 = 1 + r^1_1 - r^2_2 - r^3_3, \quad 4\lambda_2^2 = 1 + r^2_2 - r^1_1 - r^3_3, \quad 4\lambda_3^2 = 1 + r^3_3 - r^1_1 - r^2_2 \tag{2}$$

Let us further consider the diagonal elements of the matrix R and using the unity condition of the quaternion λ we write

$$\begin{aligned} r^1_1 &= 1 - 2(\lambda_2^2 + \lambda_3^2) = 1 - 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + 2\lambda_1^2 = 1 - 2(1 - \lambda_0^2) + 2\lambda_1^2 = -1 + 2\lambda_0^2 + 2\lambda_1^2, \\ r^2_2 &= 1 - 2(\lambda_1^2 + \lambda_3^2) = 1 - 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + 2\lambda_2^2 = 1 - 2(1 - \lambda_0^2) + 2\lambda_2^2 = -1 + 2\lambda_0^2 + 2\lambda_2^2, \\ r^3_3 &= 1 - 2(\lambda_1^2 + \lambda_2^2) = 1 - 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + 2\lambda_3^2 = 1 - 2(1 - \lambda_0^2) + 2\lambda_3^2 = -1 + 2\lambda_0^2 + 2\lambda_3^2. \end{aligned}$$

The second group of formulas we need is obtained:

$$2\lambda_1^2 = r_1^1 - 2\lambda_0^2 + 1, \quad 2\lambda_2^2 = r_2^2 - 2\lambda_0^2 + 1, \quad 2\lambda_3^2 = r_3^3 - 2\lambda_0^2 + 1. \tag{3}$$

Formulas (3) allow us to find the largest absolute value component λ_1, λ_2, λ_3 of the vector part of a quaternion using the elements of the matrix and λ_0.

We obtain the last set of formulas by summing the symmetric elements of the matrix R:

$$\begin{aligned}
r_1^2 + r_2^1 &= 2\lambda_0\lambda_3 + 2\lambda_1\lambda_2 + 2\lambda_1\lambda_2 - 2\lambda_0\lambda_3 = 4\lambda_1\lambda_2,\\
r_3^1 + r_1^3 &= 2\lambda_1\lambda_3 + 2\lambda_0\lambda_2 + 2\lambda_1\lambda_3 - 2\lambda_0\lambda_2 = 4\lambda_1\lambda_3,\\
r_2^3 + r_3^2 &= 2\lambda_2\lambda_3 - 2\lambda_0\lambda_1 + 2\lambda_2\lambda_3 + 2\lambda_0\lambda_1 = 4\lambda_2\lambda_3,
\end{aligned}$$

we obtain the third, final group of necessary formulas:

$$4\lambda_1\lambda_2 = r_1^2 + r_2^1, 4\lambda_1\lambda_3 = r_3^1 + r_1^3, 4\lambda_2\lambda_3 = r_2^3 + r_3^1. \tag{4}$$

Now we have the entire necessary set of formulas at our disposal and we can move on to the presentation of the algorithm itself.

- If $|\lambda_0| \leqslant 1/2$, then dividing by λ_0 when calculating λ_1, λ_2, λ_3 using the set of formulas (1) can lead to an accumulation of errors, so it is more correct to start calculating from the largest component of the vector part.
- To find out which of the components λ_1, λ_2, λ_3 is larger, use the formulas (3). From the formulas it is clear that there is no need to calculate the entire expression; it is sufficient to compare the components of the matrix r_1^1, r_2^2, and r_3^3.
- If $r_1^1 > r_2^2$ and $r_1^1 > r_3^3$, then the component λ_1 is the largest. We calculate it using the formula from (2), λ_2 and λ_3 using the formulas (4), and λ_0 using the formula from (1), i.e.

$$\lambda_1 = \pm\frac{1}{2}\sqrt{1 + r_1^1 - r_2^2 - r_3^3}, \quad \lambda_2 = \frac{r_1^2 + r_2^1}{4\lambda_1}, \quad \lambda_3 = \frac{r_3^1 + r_1^3}{4\lambda_1}, \quad \lambda_0 = \frac{r_2^3 - r_3^2}{4\lambda_1}.$$

- Otherwise, if $r_2^2 > r_3^3$, then λ_2 is the largest, then

$$\lambda_2 = \pm\frac{1}{2}\sqrt{1 + r_2^2 - r_1^1 - r_3^3}, \quad \lambda_1 = \frac{r_1^2 + r_2^1}{4\lambda_2}, \quad \lambda_3 = \frac{r_2^3 + r_3^1}{4\lambda_2}, \quad \lambda_0 = \frac{r_3^1 - r_1^3}{4\lambda_2}.$$

- Otherwise, the only option left is when the largest is λ_3, then

$$\lambda_3 = \pm\frac{1}{2}\sqrt{1 + r_3^3 - r_1^1 - r_2^2}, \quad \lambda_1 = \frac{r_3^1 + r_1^3}{4\lambda_3}, \quad \lambda_2 = \frac{r_2^3 + r_3^1}{4\lambda_3}, \quad \lambda_0 = \frac{r_1^2 - r_2^1}{4\lambda_3}.$$

1.5. Reflections using quaternions

1.5.1. Reflection about a plane passing through the origin

Vector formula Consider a point P with radius vector $\mathbf{p}$ and a plane π passing through the origin O with unit normal vector $\mathbf{n}$, where $\mathbf{n} \perp \pi$. Relative to the vector $\mathbf{n}$, the vector $\mathbf{p}$ splits into two components $\mathbf{p}_{\|\mathbf{n}}$ and $\mathbf{p}_{\perp\mathbf{n}}$:

$$\mathbf{p} = \mathbf{p}_{\|\mathbf{n}} + \mathbf{p}_{\perp\mathbf{n}}.$$

Reflection with respect to the π plane affects only the vector $\mathbf{p}_{\|\mathbf{n}}$, while leaving the vector $\mathbf{p}_{\perp\mathbf{n}}$ unchanged. However, the vector $\mathbf{p}_{\|\mathbf{n}}$ changes sign upon reflection. As a result, the reflected vector $\mathbf{p}'$ is calculated as follows:

$$\mathbf{p}' = \mathbf{p}_{\perp\mathbf{n}} - \mathbf{p}_{\|\mathbf{n}},$$

where $\mathbf{p}_{\|\mathbf{n}} = (\mathbf{p}, \mathbf{n})\mathbf{n}$ and $\mathbf{p}_{\perp\mathbf{n}} = \mathbf{p} - \mathbf{p}_{\|\mathbf{n}} = \mathbf{p} - (\mathbf{p}, \mathbf{n})\mathbf{n}$, therefore

$$\mathbf{p}' = \underbrace{\mathbf{p} - (\mathbf{p}, \mathbf{n})\mathbf{n}}_{\mathbf{p}_{\perp\mathbf{n}}} - \underbrace{(\mathbf{p}, \mathbf{n})\mathbf{n}}_{\|\mathbf{n}} = \mathbf{p} - 2(\mathbf{p}, \mathbf{n})\mathbf{n}. \qquad (5)$$

The same formula can be written in matrix form:

$$\mathbf{p}' = (\mathrm{I} - 2\mathbf{n}\mathbf{n}^T)\mathbf{p}.$$

Quaternion formula If we associate with the radius vector $\mathbf{p}$ a pure quaternion $p = 0 + \mathbf{p}$, then the quaternion formula for reflection can be obtained from the equalityby setting $\mathbf{q} = \mathbf{n}$ and applying it to (5) from right to left:

$$\mathbf{n}\mathbf{p}\mathbf{n} = \|\mathbf{n}\|^2\mathbf{p} - 2(\mathbf{p}, \mathbf{n})\mathbf{n} = \mathbf{p} - 2(\mathbf{p}, \mathbf{n})\mathbf{n} = \mathbf{p}'.$$

Above, we associated a point with a quaternion of the general form $p = p_0 + p_x\mathrm{i} + p_y\mathrm{j} + p_z\mathrm{k}$, so in the quaternion formula for reflecting a point, we would like to see a quaternion p with a nonzero scalar part. However, the formula $\mathbf{n}p\mathbf{n}$ will give an incorrect result:

$$\mathbf{n}p\mathbf{n} = \mathbf{n}(p_0 + \mathbf{p})\mathbf{n} = \mathbf{n}\mathbf{n}p_0 + \mathbf{n}\mathbf{p}\mathbf{n} = -p_0 + \mathbf{p}',$$

where the scalar part becomes negative, although it should remain positive.

To eliminate this discrepancy, we change the formula as follows:

$$p' = \mathbf{n}(p - 2p_0)\mathbf{n} = \mathbf{n}(p_0 + \mathbf{p} - 2p_0)\mathbf{n} = \mathbf{n}(\mathbf{p} - p_0)\mathbf{n} = -\mathbf{n}\mathbf{n}p_0 + \mathbf{n}\mathbf{p}\mathbf{n} = p_0 + \mathbf{p}'.$$

If we additionally note that $p - 2p_0 = -p_0 + \mathbf{p} = -(p_0 - \mathbf{p}) = -p^*$, then the quaternion reflection formula can be written in its final form as follows:

$$p' = -\mathbf{n}p^*\mathbf{n}$$

1.5.2. Reflection relative to an arbitrary plane

Vector formula Consider an arbitrary plane π, with a unit normal vector $\mathbf{n}$, located at a distance $-d$ from the origin O. Such a plane is given by the equation:

$$(\mathbf{q}, \mathbf{n}) + d = 0,$$

where $\mathbf{q}$ is the radius vector of an arbitrary point Q belonging to the π plane. The radius vector of the projection of the origin onto the plane is calculated as $\mathbf{OO}_\perp = -d\mathbf{n}$.

The reflection of a certain point P with a radius vector $\mathbf{p}$ is carried out in three stages:

1. transferring the origin to a point on the plane by subtracting the vector $\mathbf{OO}_\perp$ from $\mathbf{p}$;
2. reflection using the formula (5);
3. returning the origin by adding the resulting vector to $\mathbf{OO}_\perp$.

Combining all three actions into one formula, we write:

$$\mathbf{p}' = \mathbf{p} + d\mathbf{n} - 2(\mathbf{p} + d\mathbf{n}, \mathbf{n})\mathbf{n} - d\mathbf{n} = \mathbf{p} - 2(\mathbf{p}, \mathbf{n})\mathbf{n} - 2d(\mathbf{n}, \mathbf{n})\mathbf{n} = \mathbf{p} - 2(\mathbf{p}, \mathbf{n})\mathbf{n} - 2d\mathbf{n}$$

and as a result we get:

$$\mathbf{p}' = \mathbf{p} - 2(\mathbf{p}, \mathbf{n})\mathbf{n} - 2d\mathbf{n}. \qquad (6)$$

It is worth paying attention to the minus sign in the formula $\mathbf{OO}_\perp = -d\mathbf{n}$, due to which, when subtracting the vector $\mathbf{OO}_\perp$ in the calculations, addition with $d\mathbf{n}$ occurs, and when adding, on the contrary, subtraction occurs.

Table 1

Kotelnikov–Study Transfer Principle

Radius vector **p**	Screw **L**
Angle θ	Dual angle Θ
Real number λ	Dual number Λ

Quaternion formula Similarly, it can be written in quaternion form:

$$\mathbf{p}' = \mathbf{p} - 2(\mathbf{p}, \mathbf{n})\mathbf{n} - 2d\mathbf{n}.$$

To prove the formula, we perform some transformations

$$p' = -\mathbf{n}(p + d\mathbf{n})^*\mathbf{n} - d\mathbf{n} = -\mathbf{n}p^*\mathbf{n} - \mathbf{n}d\mathbf{n}^*\mathbf{n} - d\mathbf{n} = -\mathbf{n}p^*\mathbf{n} - d\mathbf{n} - d\mathbf{n} =$$
$$= -\mathbf{n}(1 - \mathbf{p})\mathbf{n} - 2d\mathbf{n} = -\mathbf{n}\mathbf{n} + \mathbf{n}\mathbf{p}\mathbf{n} = 1 + \mathbf{n}\mathbf{p}\mathbf{n} - 2d\mathbf{n}.$$

Since $\mathbf{n}\mathbf{p}\mathbf{n} = \|\mathbf{n}\|^2\mathbf{p} - 2(\mathbf{p}, \mathbf{n})\mathbf{n} = \mathbf{p} - 2(\mathbf{p}, \mathbf{n})\mathbf{n}$, we can finally write

$$p' = 1 + \mathbf{p} - 2(\mathbf{p}, \mathbf{n})\mathbf{n} - 2d\mathbf{n} = 1 + \mathbf{p}',$$

from which it is clear that the vector part completely coincides with the formula (6).

2. Description of screw motion and reflection using dual quaternions

2.1. Kotelnikov–Study's transfer principle

Transfer principle. All formulas of the theory of finite rotations and the kinematics of motion of a rigid body with one fixed point, when replacing real quantities with dual analogs, are transformed into formulas of the theory of finite displacements and the kinematics of motion of a free rigid body [10, p. 67].

The principle was formulated by Alexander Petrovich Kotelnikov and Eduard Study (Eduard Study) [5, pp. 12–13].

In other words, if in the formulas for the rotation of a point in space we replace real numbers, vectors, angles and quaternions with dual numbers, screws, dual angles and dual quaternions, then we obtain the correct formulas for screw motion (table 1).

If the formulas for rotations in space are applied to **affine points** (radius vectors), then the formulas obtained by the principle of transfer should be applied to **screws**, that is, to **lines** in space.

2.2. Application of the Kotelnikov–Study transfer principle

2.2.1. Obtaining a dual quaternion of screw motion

Let's apply the Kotelnikov–Study transfer principle to derive dual quaternion formulas for screw motion. First, we write down the necessary quaternion formulas. Rotational motion around an axis passing through the origin defines a unit quaternion, which is most conveniently written in trigonometric form:

$$\lambda = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{a}, \quad \mathbf{a} = a_x\mathrm{i} + a_y\mathrm{j} + a_z\mathrm{k},$$

where θ is the angle of rotation around the axis with unit direction vector $\mathbf{a} = (a_x, a_y, a_z)^T$, $\|\mathbf{a}\| = 1$. The rotation of an affine point P, represented in homogeneous coordinates by the quaternion $p = 1 + x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$, is given by the sandwich formula:

$$p' = \lambda p \lambda^*,$$

where the new position of the point is expressed by the quaternion $p' = 1 + x'\mathrm{i} + y'\mathrm{j} + z'\mathrm{k}$.

Note also that the quaternion p need not have a unit scalar part and can define a projective point of any form, since the sandwich formula leaves the coordinate w unchanged due to the unity of $\lambda\lambda^* = 1$:

$$\lambda(w + \mathbf{p})\lambda^* = \lambda w \lambda^* + \lambda \mathbf{p} \lambda^* = w\lambda\lambda^* + \lambda\mathbf{p}\lambda^* = w + \lambda\mathbf{p}\lambda^*.$$

The scalar part w can also be equal to zero, in which case the formula defines the rotation of a point at infinity in projective space or an equivalent free vector in Cartesian space.

According to the translation principle, dual quaternion defining screw motion (translation + rotation) is obtained from a rotational quaternion by the following substitution:

- $\theta \longrightarrow \Theta = \theta + \theta^o\varepsilon$ — the angle is replaced by its dual angle;
- $\mathbf{a} \longrightarrow \mathbf{A} = \mathbf{a} + \mathbf{a}^o\varepsilon$ — the vector is replaced by a pure dual quaternion (a screw);
- $* \longrightarrow \dagger$ — the quaternion conjugate $*$ is replaced by the dual quaternion conjugate $\dagger$.

As a result of such a replacement, the dual quaternion of screw motion will be written as follows:

$$\Lambda = \cos\frac{\Theta}{2} + \sin\frac{\Theta}{2}\mathbf{A}, \quad \Theta = \theta + \theta^o\varepsilon, \quad \mathbf{A} = \mathbf{a} + \mathbf{a}^o\varepsilon. \tag{7}$$

A pure dual quaternion $\mathbf{A}$ defines an arbitrary axis of rotation with a direction unit vector $\mathbf{a}$ and a moment $\mathbf{a}^o$. The set of vectors $\{\mathbf{a} \mid \mathbf{a}^o\}$ are the Plücker coordinates, for which th Plücker condition $(\mathbf{a}, \mathbf{a}^o) = 0$ must be satisfied. The Plücker condition and the condition $\|\mathbf{a}\| = 1$ guarantee the unity of the pure dual quaternion $\mathbf{A}$, since $|\mathbf{A}| = \mathbf{A}\mathbf{A}^* = (\mathbf{a} + \mathbf{a}^o\varepsilon)(-\mathbf{a} - \mathbf{a}^o\varepsilon) = \|\mathbf{a}^2 + 2(\mathbf{a}, \mathbf{a}^o)\varepsilon\| = \|\mathbf{a}\|^2 = 1$.

The dual angle Θ specifies both the magnitude of the rotation angle θ around the axis $\mathbf{A}$ and the translation distance θ^o along the same axis $\mathbf{A}$. Trigonometric functions of the dual angle are calculated using the formulas

$$\sin\Theta = \sin(\theta + \varepsilon\theta^o) = \sin\theta + \theta^o\cos\theta\varepsilon, \quad \cos\Theta = \cos(\theta + \theta^o\varepsilon) = \cos\theta - \theta^o\sin\theta\varepsilon.$$

Using these formulas, we replace the dual number Θ in the formula (7) with the real numbers θ and θ^o and transform the dual quaternion Λ as follows:

$$\begin{aligned}\Lambda = \cos\frac{\Theta}{2} + \sin\frac{\Theta}{2}\mathbf{A} = \cos\frac{\theta}{2} - \frac{\theta^o}{2}\sin\frac{\theta}{2}\varepsilon + \left(\sin\frac{\theta}{2} + \frac{\theta^2}{2}\cos\frac{\theta}{2}\varepsilon\right)(\mathbf{a} + \mathbf{a}^o\varepsilon) = \\ = \cos\frac{\theta}{2} - \frac{\theta^o}{2}\sin\frac{\theta}{2}\varepsilon + \sin\frac{\theta}{2}\mathbf{a} + \frac{\theta^o}{2}\cos\frac{\theta}{2}\varepsilon\mathbf{a} + \sin\frac{\theta}{2}\mathbf{a}^o\varepsilon + \frac{\theta^o}{2}\cos\frac{\theta}{2}\mathbf{a}^o\varepsilon^2 = \\ = \cos\frac{\theta}{2} - \frac{\theta^o}{2}\sin\frac{\theta}{2}\varepsilon + \sin\frac{\theta}{2}\mathbf{a} + \frac{\theta^o}{2}\cos\frac{\theta}{2}\mathbf{a}\varepsilon + \sin\frac{\theta}{2}\mathbf{a}^o\varepsilon.\end{aligned}$$

Having grouped separately the terms with θ^o and without θ^o, we write the formula in the following form:

$$\Lambda = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon) + \left(\cos\frac{\theta}{2}\mathbf{a} - \sin\frac{\theta}{2}\right)\frac{\theta^o}{2}\varepsilon. \tag{8}$$

This notation has the advantage of allowing us to distinguish between pure rotation and pure translation:

- for $\theta^o = 0$, we obtain pure rotation around an arbitrary axis $\mathbf{A}$ defined by the dual quaternion $R = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon)$;
- for $\theta = 0$, the trigonometric functions take the values $\sin 0 = 0$ and $\cos 0 = 1$, and we obtain pure translation along the axis $\mathbf{A}$ defined by the dual quaternion $T = 1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon$.

2.2.2. Conjugate dual quaternion of screw motion

Now let's find the conjugate dual quaternion Λ. Recall that dual quaternion conjugation is introduced in three different ways:

- $Q^* = (q + q^o\varepsilon)^* = q^* + q^{o*}\varepsilon$ is the quaternion conjugation, which can also be called complex conjugation.
- $\overline{Q} = \overline{q + q^o\varepsilon} = q - q^o\varepsilon$ is the dual conjugation.
- $Q^\dagger = (\overline{q + q^o\varepsilon})^* = q^* - q^{o*}\varepsilon$ is the dual quaternion conjugation.

We need to calculate the dual conjugation of the dual quaternion Λ:

$$\Lambda^\dagger = \overline{\cos\Theta/2} + \overline{\sin\Theta/2}\mathbf{A}^\dagger, \quad \mathbf{A}^\dagger = \mathbf{a}^* - \mathbf{a}^{o*}\varepsilon = -\mathbf{a} + \mathbf{a}^o\varepsilon = -(\mathbf{a} - \mathbf{a}^o\varepsilon).$$

Next, we can use formulas for trigonometric functions of dual numbers:

$$\overline{\cos\Theta/2} = \cos\frac{\theta}{2} + \frac{\theta^o}{2}\sin\frac{\theta}{2}\varepsilon, \quad \overline{\sin\Theta/2} = \sin\frac{\theta}{2} - \frac{\theta^o}{2}\cos\frac{\theta}{2}\varepsilon.$$

We can, however, obtain this formula differently by writing Λ in quaternion form:

$$\Lambda = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{a} + \left(-\frac{\theta^o}{2}\sin\frac{\theta}{2} + \frac{\theta^o}{2}\cos\frac{\theta}{2}\mathbf{a} + \sin\frac{\theta}{2}\mathbf{a}^o\right)\varepsilon = \lambda + \lambda^o\varepsilon,$$

where $\lambda = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{a}$ and $\lambda^o = -\frac{\theta^o}{2}\sin\frac{\theta}{2} + \frac{\theta^o}{2}\cos\frac{\theta}{2}\mathbf{a} + \sin\frac{\theta}{2}\mathbf{a}^o$ are quaternions. Then we calculate:

$$\begin{aligned}\Lambda^\dagger = \lambda^* - \lambda^{o*}\varepsilon &= \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\mathbf{a}\right) - \left(-\frac{\theta^o}{2}\sin\frac{\theta}{2} - \frac{\theta^o}{2}\cos\frac{\theta}{2}\mathbf{a} - \sin\frac{\theta}{2}\mathbf{a}^o\right)\varepsilon = \\ &= \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\mathbf{a}\right) + \left(\frac{\theta^o}{2}\sin\frac{\theta}{2} + \frac{\theta^o}{2}\cos\frac{\theta}{2}\mathbf{a} + \sin\frac{\theta}{2}\mathbf{a}^o\right)\varepsilon = \\ &= \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a} - \mathbf{a}^o\varepsilon)\right) + \left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\mathbf{a}\right)\frac{\theta^o}{2}\varepsilon.\end{aligned}$$

Accordingly, the final formula for the conjugate dual quaternion will look like this:

$$\Lambda^\dagger = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a} - \mathbf{a}^o\varepsilon) + \left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\mathbf{a}\right)\frac{\theta^o}{2}\varepsilon.$$

2.2.3. Proof of the unity of the dual quaternion of screw motion

Let us also recall the definition of a unit dual quaternion: a dual quaternion Q is called unit dual quaternion if its modulus is equal to 1, that is, $|Q|^2 = QQ^* = |q|^2 + 2(q, q^o)\varepsilon = 1$. Let us check that Λ is unit dual quaternion, for which it is necessary to calculate the quaternion conjugate. Let us do this and at the same time note that the expression for Λ^* will differ from $\Lambda^\dagger$.

$$\Lambda^* = \lambda^* + \lambda^{o*}\varepsilon = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\mathbf{a} + \left(-\frac{\theta^o}{2}\sin\frac{\theta}{2} - \frac{\theta^o}{2}\cos\frac{\theta}{2}\mathbf{a} - \sin\frac{\theta}{2}\mathbf{a}^o\right)\varepsilon =$$

$$= \cos\frac{\theta}{2} - \frac{\theta^o}{2}\sin\frac{\theta}{2}\varepsilon - \sin\frac{\theta}{2}\mathbf{a} - \frac{\theta^o}{2}\cos\frac{\theta}{2}\mathbf{a}\varepsilon - \sin\frac{\theta}{2}\mathbf{a}^o\varepsilon - \frac{\theta^o}{2}\cos\frac{\theta}{2}\mathbf{a}^o\varepsilon^2 =$$

$$= \left(\cos\frac{\theta}{2} - \frac{\theta^o}{2}\sin\frac{\theta}{2}\varepsilon\right) - \left[\left(\sin\frac{\theta}{2} + \frac{\theta^o}{2}\cos\frac{\theta}{2}\varepsilon\right)\mathbf{a} + \left(\sin\frac{\theta}{2} + \frac{\theta^o}{2}\cos\frac{\theta}{2}\varepsilon\right)\mathbf{a}^o\varepsilon\right] =$$

$$= \left(\cos\frac{\theta}{2} - \frac{\theta^o}{2}\sin\frac{\theta}{2}\varepsilon\right) - \left(\sin\frac{\theta}{2} + \frac{\theta^o}{2}\cos\frac{\theta}{2}\varepsilon\right)(\mathbf{a} + \mathbf{a}^o\varepsilon) = \cos\frac{\Theta}{2} - \sin\frac{\Theta}{2}\mathbf{A},$$

where $\mathbf{A} = \mathbf{a} + \mathbf{a}^o\varepsilon$. Finally:

$$\Lambda^* = \left(\cos\frac{\Theta}{2} + \sin\frac{\Theta}{2}\mathbf{A}\right)^* = \cos\frac{\Theta}{2} - \sin\frac{\Theta}{2}\mathbf{A},$$

since $\mathbf{A}^* = -\mathbf{a} - \mathbf{a}^o\varepsilon = -\mathbf{A}$.

Now we can find the modulus of the dual quaternion $|\Lambda|^2 = \Lambda\Lambda^*$:

$$\Lambda\Lambda^* = \left(\cos\frac{\Theta}{2} + \sin\frac{\Theta}{2}\mathbf{A}\right)\left(\cos\frac{\Theta}{2} - \sin\frac{\Theta}{2}\mathbf{A}\right) =$$

$$= \cos^2\frac{\Theta}{2} - \cos\frac{\Theta}{2}\sin\frac{\Theta}{2}\mathbf{A} + \sin\frac{\Theta}{2}\mathbf{A}\cos\frac{\Theta}{2} - \sin\frac{\Theta}{2}\mathbf{A}\sin\frac{\Theta}{2}\mathbf{A}.$$

Recall that the multiplication of dual numbers is commutative $(a + b\varepsilon)(c + d\varepsilon) = (c + d\varepsilon)(a + b\varepsilon)$, and the multiplication of a dual number by a pure dual quaternion is also commutative: $(a + b\varepsilon)(\mathbf{a} + \mathbf{b}\varepsilon) = (\mathbf{a} + \mathbf{b}\varepsilon)(a + b\varepsilon)$. This allows us to write the expression $\Lambda\Lambda^*$ in the following form:

$$\Lambda\Lambda^* = \cos^2\frac{\Theta}{2} - \cos\frac{\Theta}{2}\sin\frac{\Theta}{2}\mathbf{A} + \sin\frac{\Theta}{2}\cos\frac{\Theta}{2}\mathbf{A} - \sin^2\frac{\Theta}{2}\mathbf{A}\mathbf{A} = \cos^2\frac{\Theta}{2} - \sin^2\frac{\Theta}{2}\mathbf{A}\mathbf{A}.$$

Let us calculate the product of pure dual quaternions $\mathbf{AA}$:

$$\mathbf{AA} = (\mathbf{a} + \mathbf{a}^o\varepsilon)(\mathbf{a} + \mathbf{a}^o\varepsilon) = \mathbf{aa} + \mathbf{aa}^o\varepsilon + \mathbf{a}^o\mathbf{a}\varepsilon + \mathbf{a}^o\mathbf{a}^o\varepsilon^2 = \mathbf{aa} + (\mathbf{aa}^o + \mathbf{a}^o\mathbf{a})\varepsilon.$$

For further simplification, we use the rule of multiplication of pure quaternions $\mathbf{pq} = -(\mathbf{p}, \mathbf{q}) + \mathbf{p} \times \mathbf{q}$, then $\mathbf{aa} = -(\mathbf{a}, \mathbf{a}) = -\|\mathbf{a}\|^2 = -1$ since $\|\mathbf{a}\| = 1$ by condition. In view of the fact that $\mathbf{aa}^o = -(\mathbf{a}, \mathbf{a}^o) + \mathbf{a} \times \mathbf{a}^o$ and $\mathbf{a}^o\mathbf{a} = -(\mathbf{a}^o, \mathbf{a}) + \mathbf{a}^o \times \mathbf{a}$ the dual part of $\mathbf{AA}$ is simplified:

$$\mathbf{aa}^o + \mathbf{a}^o\mathbf{a} = -2(\mathbf{a}, \mathbf{a}^o) + \mathbf{a} \times \mathbf{a}^o - \mathbf{a} \times \mathbf{a}^o = -2(\mathbf{a}, \mathbf{a}^o).$$

In addition, by the Plücker condition $(\mathbf{a}, \mathbf{a}^o) = 0$, because $\mathbf{a} \perp \mathbf{a}^o$, therefore $\mathbf{AA} = -1$ subject to $||\mathbf{a}|| = 1$. We obtain that $\Lambda\Lambda^* = \cos^2\frac{\Theta}{2} + \sin^2\frac{\Theta}{2}$. We use the formula $(a + b\varepsilon)^2 = a^2 + 2ab\varepsilon$ to calculate $\cos^2\Theta/2$ and $\sin^2\Theta/2$, for them the following will be true:

$$\cos^2\frac{\Theta}{2} = \left(\cos\frac{\theta}{2} - \frac{\theta^o}{2}\sin\frac{\theta}{2}\,\varepsilon\right)^2 = \cos^2\frac{\theta}{2} - 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\frac{\theta^o}{2}\varepsilon,$$

$$\sin^2\frac{\Theta}{2} = \left(\sin\frac{\theta}{2} + \frac{\theta^o}{2}\cos\frac{\theta}{2}\,\varepsilon\right)^2 = \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\frac{\theta^o}{2}\varepsilon.$$

Hence:

$$\cos^2\frac{\Theta}{2} + \sin^2\frac{\Theta}{2} = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} - 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\frac{\theta^o}{2}\varepsilon + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\frac{\theta^o}{2}\varepsilon =$$

$$= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1 \Rightarrow \cos^2\frac{\Theta}{2} + \sin^2\frac{\Theta}{2} = 1,$$

where Θ is the dual angle.

As a result, we have proved the unity of the dual quaternion $\Lambda = \cos\frac{\Theta}{2} + \sin\frac{\Theta}{2}\mathbf{A}$. It should be especially noted that an important condition is the unity of the pure quaternion $\mathbf{a}$, i.e. $\|\mathbf{a}\| = 1$ and the fulfillment of the Plücker condition $(\mathbf{a}, \mathbf{a}^o) = 0$. Without these two conditions, Λ will not be unity.

2.3. Screw motion of a point and a vector

2.3.1. Rotation around an arbitrary axis without translation

Let us consider the dual quaternion representation of an affine point P:

$$P = 1 + \mathbf{p}^o\varepsilon = p + p^o,$$

where $p^o = \mathbf{p}^o$ is a pure quaternion (radius vector), and $p = 1$ is a scalar quaternion (point O). Let's now construct a sandwich operator from the dual quaternion R:

$$R = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon).$$

where $\mathbf{A} = \mathbf{a} + \mathbf{a}^o\varepsilon$ is a pure dual quaternion defining the axis of rotation, $\mathbf{a}$ is the direction vector of the axis of rotation, $\mathbf{a}^o$ is the moment of the axis of rotation, and θ is the actual angle of rotation about the axis.

Once again, we require two important conditions to be satisfied:

1. vector $\mathbf{a}$ must be the unit vector $\|\mathbf{a}\| = 1$;
2. vectors $\mathbf{a}$ and $\mathbf{a}^o$ satisfy the Plücker condition $(\mathbf{a}, \mathbf{a}^o) = 0$.

Let's find the conjugate dual quaternion $R^\dagger = (\bar{R})^* = \overline{(R^*)}$:

$$R^* = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\mathbf{a} - \sin\frac{\theta}{2}\mathbf{a}^o\varepsilon \quad R^\dagger = \overline{(R^*)} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\mathbf{a} + \sin\frac{\theta}{2}\mathbf{a}^o\varepsilon = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a} - \mathbf{a}^o\varepsilon).$$

The final formula for the sandwich operator will look like this:

$$P' = RPR^\dagger = R(1 + \mathbf{p}^o\varepsilon)R^\dagger = RR^\dagger + R\mathbf{p}^o\varepsilon R^\dagger.$$

Let's first find the dual quaternion product $RR^\dagger$:

$$\begin{aligned} RR^\dagger &= \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon)\right)\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a} - \mathbf{a}^o\varepsilon)\right) = \\ &= \cos^2\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}(\mathbf{a} - \mathbf{a}^o\varepsilon) + \sin\frac{\theta}{2}\cos\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon) - \sin^2\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon)(\mathbf{a} - \mathbf{a}^o\varepsilon) = \\ &= \cos^2\frac{\theta}{2} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon - \mathbf{a} + \mathbf{a}^o\varepsilon) + \sin^2\frac{\theta}{2}(1 + 2\mathbf{a} \times \mathbf{a}^o\varepsilon) = \\ &= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathbf{a}^o\varepsilon + 2\sin^2\frac{\theta}{2}\mathbf{a} \times \mathbf{a}^o\varepsilon = \\ &= 1 + \sin\theta\mathbf{a}^o\varepsilon + (1 - \cos\theta)\mathbf{a} \times \mathbf{a}^o\varepsilon = 1 + (\sin\theta\mathbf{a}^o + (1 - \cos\theta)\mathbf{a} \times \mathbf{a}^o)\varepsilon. \end{aligned}$$

Note that to simplify the product of vectors $(\mathbf{a} + \mathbf{a}^o\varepsilon)(\mathbf{a} - \mathbf{a}^o\varepsilon)$, the following calculations were performed:

$$(\mathbf{a} + \mathbf{a}^o\varepsilon)(\mathbf{a} - \mathbf{a}^o\varepsilon) = \mathbf{aa} - \mathbf{aa}^o\varepsilon + \mathbf{a}^o\mathbf{a}\varepsilon - \mathbf{a}^o\mathbf{a}^o\varepsilon\varepsilon = \mathbf{aa} + (\mathbf{a}^o\mathbf{a} - \mathbf{aa}^o)\varepsilon,$$

where $\mathbf{aa} = -(\mathbf{a}, \mathbf{a}) + \mathbf{a} \times \mathbf{a} = -\|\mathbf{a}\|^2 = -1$, therefore $\mathbf{aa} = -1$. The subtraction was simplified as follows:

$$\mathbf{a}^o\mathbf{a} - \mathbf{aa}^o = -(\mathbf{a}^o, \mathbf{a}) + \mathbf{a}^o \times \mathbf{a} + (\mathbf{a}, \mathbf{a}^o) - \mathbf{a} \times \mathbf{a}^o = -2\mathbf{a} \times \mathbf{a}^o.$$

The result of simplifications is an expression of the form:

$$(\mathbf{a} + \mathbf{a}^o\varepsilon)(\mathbf{a} - \mathbf{a}^o\varepsilon) = (-1 - 2\mathbf{a} \times \mathbf{a}^o\varepsilon).$$

As a result, we obtain the formula for the dual quaternion product $RR^\dagger$:

$$RR^\dagger = 1 + (\sin\theta\mathbf{a}^o + (1-\cos\theta)\mathbf{a}\times\mathbf{a}^o)\varepsilon, \tag{9}$$

where $\mathbf{a}\times\mathbf{a}^o$ has the geometric meaning of the projection of the origin onto the **A** axis or, in other words, the point of the axis closest to the origin.

Now we calculate the second term $R\mathbf{p}^o\varepsilon R^\dagger$:

$$\begin{aligned}
R\mathbf{p}^o\varepsilon R^\dagger &= \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a}+\mathbf{a}^o\varepsilon)\right)\mathbf{p}^o\varepsilon\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a}-\mathbf{a}^o\varepsilon)\right) = \\
&= \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a}+\mathbf{a}^o\varepsilon)\right)\left(\cos\frac{\theta}{2}\mathbf{p}^o\varepsilon - \sin\frac{\theta}{2}\mathbf{p}^o\mathbf{a}\varepsilon + \sin\frac{\theta}{2}\mathbf{p}^o\mathbf{a}^o\varepsilon^2\right) = \\
&= \cos^2\frac{\theta}{2}\mathbf{p}^o\varepsilon - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathbf{p}^o\mathbf{a}\varepsilon + \sin\frac{\theta}{2}\cos\frac{\theta}{2}(\mathbf{a}+\mathbf{a}^o\varepsilon)\mathbf{p}^o\varepsilon - \sin^2\frac{\theta}{2}(\mathbf{a}+\mathbf{a}^o\varepsilon)\mathbf{p}^o\mathbf{a}\varepsilon = \\
&= \cos^2\frac{\theta}{2}\mathbf{p}^o\varepsilon - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathbf{p}^o\mathbf{a}\varepsilon + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathbf{a}\mathbf{p}^o\varepsilon - \sin^2\frac{\theta}{2}\mathbf{a}\mathbf{p}^o\mathbf{a}\varepsilon = \\
&= \cos^2\frac{\theta}{2}\mathbf{p}^o\varepsilon + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(\mathbf{a}\mathbf{p}^o - \mathbf{p}^o\mathbf{a}\right)\varepsilon - \sin^2\frac{\theta}{2}\mathbf{a}\mathbf{p}^o\mathbf{a}\varepsilon = \\
&= \cos^2\frac{\theta}{2}\mathbf{p}^o\varepsilon + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathbf{a}\times\mathbf{p}^o\varepsilon - \sin^2\frac{\theta}{2}\left(\mathbf{p}^o - 2(\mathbf{a},\mathbf{p}^o)\mathbf{a}\right)\varepsilon = \\
&= \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\mathbf{p}^o\varepsilon + \sin\theta\mathbf{a}\times\mathbf{p}^o\varepsilon + \sin^2\frac{\theta}{2}\cdot 2(\mathbf{a},\mathbf{p}^o)\mathbf{a}\varepsilon = \\
&= \cos\theta\mathbf{p}^o\varepsilon + \sin\theta\mathbf{a}\times\mathbf{p}^o\varepsilon + (1-\cos\theta)(\mathbf{a},\mathbf{p}^o)\mathbf{a}\varepsilon.
\end{aligned}$$

The following simplifications were used in calculating the main part:

$$(\mathbf{a}+\mathbf{a}^o\varepsilon)\mathbf{p}^o\varepsilon = \mathbf{a}\mathbf{p}^o\varepsilon + \mathbf{a}^o\mathbf{p}^o\varepsilon^2 = \mathbf{a}\mathbf{p}^o\varepsilon,$$

$$(\mathbf{a}+\mathbf{a}^o\varepsilon)\mathbf{p}^o\mathbf{a}\varepsilon = \mathbf{a}\mathbf{p}^o\mathbf{a}\varepsilon + \mathbf{a}^o\mathbf{p}^o\mathbf{a}\varepsilon^2 = \mathbf{a}\mathbf{p}^o\mathbf{a}\varepsilon,$$

$$\mathbf{a}\mathbf{p}^o - \mathbf{p}^o\mathbf{a} = -(\mathbf{a},\mathbf{p}^o) + \mathbf{a}\times\mathbf{p}^o + (\mathbf{p}^o,\mathbf{a}) - \mathbf{p}^o\times\mathbf{a} = 2\mathbf{a}\times\mathbf{p}^o,$$

$$\begin{aligned}
\mathbf{a}\mathbf{p}^o\mathbf{a} &= \mathbf{a}\left(-(\mathbf{p}^o,\mathbf{a}) + \mathbf{p}^o\times\mathbf{a}\right) = -(\mathbf{p}^o,\mathbf{a})\mathbf{a} + \mathbf{a}(\mathbf{p}^o\times\mathbf{a}) = \\
&= -(\mathbf{p}^o,\mathbf{a})\mathbf{a} - (\mathbf{a},\mathbf{p}^o\times\mathbf{a}) + \mathbf{a}\times\mathbf{p}^o\times\mathbf{a} = -(\mathbf{p}^o,\mathbf{a})\mathbf{a} + \mathbf{a}\times\mathbf{p}^o\times\mathbf{a} = \\
&= -(\mathbf{p}^o,\mathbf{a})\mathbf{a} + \mathbf{p}^o||\mathbf{a}||^2 - (\mathbf{a},\mathbf{p}^o)\mathbf{a} = \mathbf{p}^o - 2(\mathbf{a},\mathbf{p}^o)\mathbf{a}.
\end{aligned}$$

In the end we got:

$$R\mathbf{p}^o\varepsilon R^\dagger = \left(\cos\theta\mathbf{p}^o + \sin\theta\mathbf{a}\times\mathbf{p}^o + (1-\cos\theta)(\mathbf{a},\mathbf{p}^o)\mathbf{a}\right)\varepsilon.$$

Note that in this expression the moment $\mathbf{a}^o$ is missing and the expression in brackets at the imaginary unit ε exactly repeats Rodrigues' formula for the rotation of the radius vector around the axis with the direction vector **a** passing through the origin.

Let's now write down the complete formula:

$$RPR^\dagger = RR^\dagger + R\mathbf{p}^o\varepsilon R^\dagger = 1 + [\cos\theta\mathbf{p}^o + \sin\theta\mathbf{a}\times\mathbf{p}^o + (1-\cos\theta)(\mathbf{a},\mathbf{p}^o)\mathbf{a} + \sin\theta\mathbf{a}^o + (1-\cos\theta)\mathbf{a}\times\mathbf{a}^o]\varepsilon, \tag{10}$$

where $\mathbf{a}\times\mathbf{a}^o$ corresponds to a point on the axis of rotation. The part responsible for rotation around an arbitrary axis in this case is hidden in the term $RR^\dagger$.

It is also worth noting the importance of the scalar part in the dual quaternion representation of the point $P = 1 + \mathbf{p}^o\varepsilon$. Without this part, there would be no $RR^\dagger$ term in the final formula.

A direction vector (free vector) can be represented by a pure dual quaternion:

$$\mathbf{V} = 0 + \mathbf{v}^o\varepsilon$$

and the «translational» part of $RR^\dagger$ does not act on such a dual quaternion:

$$R\mathbf{V}R^\dagger = R0R^\dagger + R\mathbf{v}^o\varepsilon R^\dagger = R\mathbf{v}^o\varepsilon R^\dagger = (\cos\theta\mathbf{v}^o + \sin\theta\mathbf{a}\times\mathbf{v}^o + (1-\cos\theta)(\mathbf{a},\mathbf{v}^o)\mathbf{a})\,\varepsilon. \tag{11}$$

Similarly, a point in projective space with homogeneous coordinates $(x, y, z\,:\,w) = (\mathbf{p}^o\mid w)$ is represented by the dual quaternion $P_w = w + \mathbf{p}^o\varepsilon$ and the same formula can be used:

$$\begin{aligned}
RP_wR^\dagger &= RwR^\dagger + R\mathbf{p}^o\varepsilon R^\dagger = wRR^\dagger + R\mathbf{p}^o\varepsilon R^\dagger = \\
&= w + [\cos\theta\mathbf{p}^o + \sin\theta\mathbf{a}\times\mathbf{p}^o + (1-\cos\theta)(\mathbf{a},\mathbf{p}^o)\mathbf{a} + w\sin\theta\mathbf{a}^o + w(1-\cos\theta)\mathbf{a}\times\mathbf{a}^o]\varepsilon.
\end{aligned}$$

2.3.2. Rotation around an arbitrary axis using Rodrigues' formula

Let us show that the formulas (10) and (11) can be obtained from the usual Rodrigues vector formula. We will consider the rotation of a point using the usual vector notation. We will associate a point P with a vector $\mathbf{p}$, where the superscript o is removed since we have moved to the vector formalism.

We perform the rotation around an axis passing through point P_0 in the direction of the radius vector $\mathbf{a}$. We associate point P_0 with the radius vector $\mathbf{p}_0$. We perform the rotation using Rodrigues' formula in three steps:

1. Subtract the vector $\mathbf{p}_0$ from $\mathbf{p}$, thereby moving the origin to point P_0 or, alternatively, moving the axis to the origin of the new coordinate system.
2. We use Rodrigues' formula to perform the rotation by applying it to the vector $\mathbf{p} - \mathbf{p_0}$.
3. Add the vector p_0 to the result of the rotation, returning the coordinate system to its original position.

If $R_{\theta,\mathbf{a}}(\mathbf{p}) = \cos\theta\mathbf{p} + \sin\theta\mathbf{a}\times\mathbf{p} + (1-\cos\theta)(\mathbf{a},\mathbf{p})\mathbf{a}$, then the rotation around the axis passing through the point P_0 with the direction vector $\mathbf{a}$ will be given by the formula:

$$\mathbf{p}' = R_{\theta,\mathbf{a}}(\mathbf{p} - \mathbf{p}_0) + \mathbf{p}_0.$$

Let's expand on this formula:

$$\begin{aligned}
R_{\theta,\mathbf{a}}(\mathbf{p} - \mathbf{p_0}) + \mathbf{p}_0 &= \cos\theta(\mathbf{p} - \mathbf{p}_0) + \sin\theta\mathbf{a}\times(\mathbf{p} - \mathbf{p}_0) + (1-\cos\theta)(\mathbf{a},\mathbf{p} - \mathbf{p}_0)\mathbf{a} + \mathbf{p}_0 = \\
&= \cos\theta\mathbf{p} - \cos\theta\mathbf{p}_0 + \sin\theta\mathbf{a}\times\mathbf{p} - \sin\theta\mathbf{a}\times\mathbf{p}_0 + (1-\cos\theta)(\mathbf{a},\mathbf{p})\mathbf{a} - (1-\cos\theta)(\mathbf{a},\mathbf{p}_0)\mathbf{a} + \mathbf{p}_0 = \\
&= \cos\theta\mathbf{p} + \sin\theta\mathbf{a}\times\mathbf{p} + (1-\cos\theta)(\mathbf{a},\mathbf{p})\mathbf{a} - \cos\theta\mathbf{p}_0 - \sin\theta\mathbf{a}\times\mathbf{p}_0 - (1-\cos\theta)(\mathbf{a},\mathbf{p}_0)\mathbf{a} + \mathbf{p}_0.
\end{aligned}$$

The choice of the point P_0 on the straight axis of rotation is generally arbitrary, however, if we define the axis by Plücker coordinates using the screw $\mathbf{A} = \mathbf{a} + \mathbf{a}^o\varepsilon$, then we can choose $\mathbf{p}_0 = \mathbf{a}\times\mathbf{a}^o$ — the projection of the point O onto the straight line. With this choice of point, the expressions in the tail will be simplified:

$$\mathbf{a}\times\mathbf{p}_0 = \mathbf{a}\times\mathbf{a}\times\mathbf{a}^o = \mathbf{a}(\mathbf{a},\mathbf{a}^o) - \mathbf{a}^o(\mathbf{a},\mathbf{a}) = -\mathbf{a}^o$$

due to the Plücker condition $(\mathbf{a},\mathbf{a}^o) = 0$ and the normalization of $\mathbf{a}$ $\|\mathbf{a}\| = 1$

$$(\mathbf{a},\mathbf{p}_0)\mathbf{a} = (\mathbf{a},\mathbf{a}\times\mathbf{a}^o)\mathbf{a} = 0,$$

because $(\mathbf{a},\mathbf{a}\times\mathbf{a}^o) = 0$.

$$-\cos\theta\mathbf{p}_0 + \sin\theta\mathbf{a} + \mathbf{p}_0 = (1-\cos\theta)\mathbf{p}_0 + \sin\theta\mathbf{a}^o = (1-\cos\theta)\mathbf{a}\times\mathbf{a}^o + \sin\theta\mathbf{a}^o.$$

As a result, given that $\mathbf{p}_0 = \mathbf{a} \times \mathbf{a}^o$ we get:

$$R_{\theta,\mathbf{a}}(\mathbf{p} - \mathbf{p}_0) + \mathbf{p}_0 = \cos\theta \mathbf{p} + \sin\theta \mathbf{a} \times \mathbf{p} + (1 - \cos\theta)(\mathbf{a}, \mathbf{p})\mathbf{a} + \sin\theta \mathbf{a}^o + (1 - \cos\theta)\mathbf{a} \times \mathbf{a}^o,$$

which completely coincides with the formula (10) obtained by the dual quaternion method up to the notation $\mathbf{p} \to \mathbf{p}^o$.

2.3.3. Translation along an axis without rotation

Let's write the dual quaternion for translation along the axis $\mathbf{A} = \mathbf{a} + \mathbf{a}^o\varepsilon$:

$$T = 1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon,$$

where θ^o is the dual part of the dual angle Θ.

$$T^\dagger = \overline{(T^*)} = \overline{(1 + \frac{\theta^o}{2}\mathbf{a}^*\varepsilon)} = \overline{(1 - \frac{\theta^o}{2}\mathbf{a}\varepsilon)} = 1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon = T.$$

We get that $T^\dagger = T$.

Let's apply the sandwich product to the point $P = 1 + \mathbf{p}^o\varepsilon$:

$$\begin{aligned} P' = TPT^\dagger = TPT = \left(1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon\right)(1 + \mathbf{p}^o\varepsilon)\left(1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon\right) = \\ = \left(1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon\right)\left(1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon + \mathbf{p}^o\varepsilon\right) = 1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon + \mathbf{p}^o\varepsilon + \frac{\theta^o}{2}\mathbf{a}\varepsilon + \frac{\theta^o}{2}\frac{\theta^o}{2}\mathbf{a}\mathbf{a}\varepsilon^2 + \frac{\theta^o}{2}\mathbf{a}\mathbf{p}^o\varepsilon^2 = \\ = 1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon + \mathbf{p}^o\varepsilon + \frac{\theta^o}{2}\mathbf{a}\varepsilon = 1 + \theta^o\mathbf{a}\varepsilon + \mathbf{p}^o\varepsilon = 1 + (\theta^o\mathbf{a} + \mathbf{p}^o)\varepsilon = 1 + (\mathbf{p}^o + \theta^o\mathbf{a})\varepsilon. \end{aligned}$$

$$P' = TPT^\dagger = 1 + (\mathbf{p}^o + \theta^o\mathbf{a})\varepsilon.$$

2.3.4. Composition of rotations and translations

Above, we used the translation principle to obtain the screw motion dual quaternion and wrote it in terms of the dual (7) and real (8) angles. However, interestingly, it can be obtained by dual quaternion multiplication of only the rotational and only the translational dual quaternions.

Let us designate:

$$R = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon),$$

$$T = 1 + \frac{\theta^o}{2}n\varepsilon,$$

$$\begin{aligned} RT = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon)\right)\left(1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon\right) = \\ = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon) + \frac{\theta^o}{2}\cos\frac{\theta}{2}\mathbf{a}\varepsilon + \sin\frac{\theta}{2}\mathbf{a}\frac{\theta^o}{2}\mathbf{a}\varepsilon + \sin\frac{\theta}{2}\frac{\theta^o}{2}\mathbf{a}^o\mathbf{a}\varepsilon^2 = \\ = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon) + \cos\frac{\theta}{2}\frac{\theta^o}{2}\mathbf{a}\varepsilon - \sin\frac{\theta}{2}\frac{\theta^o}{2}\varepsilon = \\ = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon) + \left(\cos\frac{\theta}{2}\mathbf{a} - \sin\frac{\theta}{2}\right)\frac{\theta^o}{2}\varepsilon = \Lambda. \end{aligned}$$

Taking into account the simplification $\mathbf{aa} = -(\mathbf{a},\mathbf{a}) + \mathbf{a} \times \mathbf{a} = -\|\mathbf{a}\|^2 = -1$, we obtain the final formula:

$$\Lambda = RT,$$

which completely coincides with the full formula:

$$TR = \left(1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon\right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon)\right) = \cos\frac{\theta}{2} + \frac{\theta^o}{2}\mathbf{a}\cos\frac{\theta}{2}\varepsilon + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon) + \sin\frac{\theta}{2}\frac{\theta^o}{2}\mathbf{aa}\varepsilon =$$
$$= \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon) + \left(\cos\frac{\theta}{2}\mathbf{a} - \sin\frac{\theta}{2}\right)\frac{\theta^o}{2}\varepsilon = \Lambda.$$

It turns out that translation and rotation along the screw axis commute:

$$\Lambda = RT = TR.$$

2.4. Screw motion of a straight line

An arbitrary line with a specified direction $\mathbf{a}$ is defined by a pure dual quaternion $\mathbf{L} = \mathbf{v} + \mathbf{m}\varepsilon$. The components of the vectors $\{\mathbf{v} \mid \mathbf{m}\}$ are Plücker coordinates, and the Plücker condition $(\mathbf{v},\mathbf{m}) = 0$ is satisfied. Screw motion is defined by the same dual quaternion Λ, but the sandwich formula looks somewhat different:

$$\mathbf{L}' = \Lambda\mathbf{L}\Lambda^*.$$

Just as for a point, we will first consider the translation of a straight line, then rotations, and then find their composition.

2.4.1. Translation of a straight line along an axis

Consider the line $\mathbf{L} = \mathbf{v} + \mathbf{m}\varepsilon$ and apply T to it as a sandwich operator:

$$T = 1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon, \quad T^* = 1 - \frac{\theta^o}{2}\mathbf{a}\varepsilon,$$

$$T(\mathbf{v} + \mathbf{m}\varepsilon)T^* = \left(1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon\right)(\mathbf{v} + \mathbf{m}\varepsilon)\left(1 - \frac{\theta^o}{2}\mathbf{a}\varepsilon\right).$$

It can be shown that $TT^* = 1$, since $T^* = 1 - \frac{\theta^o}{2}\mathbf{a}\varepsilon$, then

$$\left(1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon\right)\left(1 - \frac{\theta^o}{2}\mathbf{a}\varepsilon\right) = 1 - \frac{\theta^o}{2}\mathbf{a}\varepsilon + \frac{\theta^o}{2}\mathbf{a}\varepsilon = 1 \Rightarrow |T| = TT^* = 1.$$

Let's now reveal the formula:

$$T(\mathbf{v} + \mathbf{m}\varepsilon)T^* = T\mathbf{v}T^* + T\mathbf{m}T^*\varepsilon.$$

Because

$$T\mathbf{v}T^* = \left(1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon\right)\left(\mathbf{v} - \frac{\theta^o}{2}\mathbf{va}\varepsilon\right) = \mathbf{v} + \frac{\theta^o}{2}\mathbf{av}\varepsilon - \frac{\theta^o}{2}\mathbf{va}\varepsilon$$

and

$$T\mathbf{m}T^* = \mathbf{m} + \frac{\theta^o}{2}\mathbf{am}\varepsilon - \frac{\theta^o}{2}\mathbf{ma}\varepsilon,$$

that

$$T\mathbf{m}T^*\varepsilon = \mathbf{m}\varepsilon + \frac{\theta^o}{2}\mathbf{am}\varepsilon^2 - \frac{\theta^o}{2}\mathbf{ma}\varepsilon^2 = \mathbf{m}\varepsilon.$$

$$T\mathbf{v}T^* + T\mathbf{m}T^*\varepsilon = \mathbf{v} + \frac{\theta^o}{2}(\mathbf{av} - \mathbf{va})\varepsilon + \mathbf{m}\varepsilon.$$

$$\mathbf{av} - \mathbf{va} = -(\mathbf{a}, \mathbf{v}) + \mathbf{a} \times \mathbf{v} + (\mathbf{v}, \mathbf{a}) - \mathbf{v} \times \mathbf{a} = 2\mathbf{a} \times \mathbf{v}.$$

Then it turns out that:

$$L' = TLT^* = \mathbf{v} + \frac{\theta^o}{2} 2\mathbf{a} \times \mathbf{v}\varepsilon + \mathbf{m}\varepsilon = \mathbf{v} + (\mathbf{m} + \theta^o \mathbf{a} \times \mathbf{v})\varepsilon,$$

where $\mathbf{m} = \mathbf{p}_0 \times \mathbf{v} \Rightarrow \mathbf{m} + \theta^o\mathbf{a} \times \mathbf{v} = \mathbf{p}_0 \times \mathbf{v} + \theta^o\mathbf{a} \times \mathbf{v} = (\mathbf{p}_0 + \theta^o\mathbf{a}) \times \mathbf{v}$, therefore we get:

$$L' = \mathbf{v} + (\mathbf{p}_0 + \theta^o\mathbf{a}) \times \mathbf{v}\varepsilon.$$

As a result, as expected, the direction vector $\mathbf{v}$ does not change during parallel translation (translation), but the moment of the line is transformed. The line was moved from the point P_0 towards the $\mathbf{a}$ axis by the amount θ^o. Note that we can map T^{-1} onto T^*, since $|T| = 1$ and $T^* = T^{-1}$. Therefore, the sandwich formula for translation can also be written as follows [13, p. 51]:

$$L' = TLT^* = TLT^{-1}.$$

2.4.2. Rotation of a line around an axis

Let's consider the rotating dual quaternion R and find its conjugate dual quaternion R^*:

$$R = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon), \quad R^* = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon).$$

It was shown earlier that $\mathbf{AA} = -1$, where $\mathbf{A} = \mathbf{a} + \mathbf{a}^o\varepsilon$, so:

$$RR^* = \cos^2\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathbf{A} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathbf{A} - \sin^2\frac{\theta}{2}\mathbf{AA} = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1 \Rightarrow RR^* = 1,$$

from which it follows that $R^{-1} = R^*$.

The rotation of the line $\mathbf{L} = \mathbf{v} + \mathbf{m}\varepsilon$ is carried out by the sandwich operator:

$$\mathbf{L}' = R\mathbf{L}R^* = R\mathbf{v}R^* + R\mathbf{m}R^*\varepsilon,$$

Let's calculate the first term in this formula:

$$\begin{aligned} R\mathbf{v}R = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{A}\right)\mathbf{v}\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\mathbf{A}\right) = \\ = \cos^2\frac{\theta}{2}\mathbf{v} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathbf{vA} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathbf{Av} - \sin^2\frac{\theta}{2}\mathbf{AvA} = \\ = \cos^2\frac{\theta}{2}\mathbf{v} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}(\mathbf{Av} - \mathbf{vA}) - \sin^2\frac{\theta}{2}\mathbf{AvA}. \end{aligned}$$

Because

$$\mathbf{Av} = (\mathbf{a} + \mathbf{a}^o\varepsilon)\mathbf{v} = \mathbf{av} + \mathbf{a}^o\mathbf{v}\varepsilon = -(\mathbf{a}, \mathbf{v}) - (\mathbf{a}^o, \mathbf{v})\varepsilon + \mathbf{a} \times \mathbf{v} + \mathbf{a}^o \times \mathbf{v}\varepsilon$$

and

$$\mathbf{vA} = \mathbf{v}(\mathbf{a} + \mathbf{a}^o\varepsilon) = \mathbf{va} + \mathbf{va}^o\varepsilon = -(\mathbf{v}, \mathbf{a}) - (\mathbf{v}, \mathbf{a}^o)\varepsilon + \mathbf{v} \times \mathbf{a} + \mathbf{v} \times \mathbf{a}^o\varepsilon,$$

that

$$\mathbf{Av} - \mathbf{vA} = 2\mathbf{a} \times \mathbf{v} + 2\mathbf{a}^o \times \mathbf{v}\varepsilon.$$

Next, we calculate the dual quaternion product $\mathbf{AvA}$:

$$\mathbf{AvA} = (\mathbf{a}+\mathbf{a}^{o}\varepsilon)\mathbf{v}(\mathbf{a}+\mathbf{a}^{o}\varepsilon) = (\mathbf{a}+\mathbf{a}^{o}\varepsilon)(\mathbf{va}+\mathbf{va}^{o}\varepsilon) = \mathbf{ava}+\mathbf{ava}^{o}\varepsilon+\mathbf{a}^{o}\mathbf{va}\varepsilon+\mathbf{a}^{o}\mathbf{va}^{o}\varepsilon^{2} = \mathbf{ava}+(\mathbf{ava}^{o}+\mathbf{a}^{o}\mathbf{va})\varepsilon,$$

where to simplify the dual part $\mathbf{ava}^{o} + \mathbf{a}^{o}\mathbf{va}$ we take into account that

$$\begin{aligned}\mathbf{ava}^{o} = -\mathbf{a}(\mathbf{v},\mathbf{a}^{o}) + \mathbf{av}\times\mathbf{a}^{o} = -(\mathbf{v},\mathbf{a}^{o})\mathbf{a} - (\mathbf{a},\mathbf{v}\times\mathbf{a}^{o}) + \mathbf{a}\times\mathbf{v}\times\mathbf{a}^{o} = \\ = -(\mathbf{v},\mathbf{a}^{o})\mathbf{a} - (\mathbf{a},\mathbf{v}\times\mathbf{a}^{o}) + \mathbf{v}(\mathbf{a},\mathbf{a}^{o}) - \mathbf{a}^{o}(\mathbf{a},\mathbf{v}),\end{aligned}$$

$$\begin{aligned}\mathbf{a}^{o}\mathbf{va} = -\mathbf{a}^{o}(\mathbf{v},\mathbf{a}) + \mathbf{a}^{o}\mathbf{v}\times\mathbf{a} = -(\mathbf{v},\mathbf{a})\mathbf{a}^{o} - (\mathbf{a}^{o},\mathbf{v}\times\mathbf{a}) + \mathbf{a}^{o}\times\mathbf{v}\times\mathbf{a} = \\ = -(\mathbf{v},\mathbf{a})\mathbf{a}^{o} - (\mathbf{a}^{o},\mathbf{v}\times\mathbf{a}) + \mathbf{v}(\mathbf{a}^{o},\mathbf{a}) - \mathbf{a}(\mathbf{a}^{o},\mathbf{v}),\end{aligned}$$

after which the expression in brackets is simplified:

$$\mathbf{ava}^{o} + \mathbf{a}^{o}\mathbf{va} = -2(\mathbf{v},\mathbf{a}^{o})\mathbf{a} - 2(\mathbf{v},\mathbf{a})\mathbf{a}^{o} + 2(\mathbf{a},\mathbf{a}^{o})\mathbf{v} = -2\big((\mathbf{v},\mathbf{a}^{o})\mathbf{a} + (\mathbf{v},\mathbf{a})\mathbf{a}^{o}\big).$$

We used the Plücker condition $(\mathbf{a}^{o},\mathbf{a}) = 0$ and the mixed product property $(\mathbf{a}^{o},\mathbf{v}\times\mathbf{a}) = -(\mathbf{a},\mathbf{v}\times\mathbf{a}^{o})$. Given that

$$\mathbf{ava} = -\mathbf{a}(\mathbf{v},\mathbf{a}) + \mathbf{av}\times\mathbf{a} = -\mathbf{a}(\mathbf{v},\mathbf{a}) - (\mathbf{a},\mathbf{v}\times\mathbf{a}) + \mathbf{a}\times\mathbf{v}\times\mathbf{a} = -\mathbf{a}(\mathbf{v},\mathbf{a}) + \mathbf{v}(\mathbf{a},\mathbf{a}) - \mathbf{a}(\mathbf{a},\mathbf{v}) = \mathbf{v} - 2\mathbf{a}(\mathbf{v},\mathbf{a}),$$

we obtain the final expression:

$$\mathbf{AvA} = \mathbf{v} - 2\mathbf{a}(\mathbf{v},\mathbf{a}) - 2\big((\mathbf{v},\mathbf{a}^{o})\mathbf{a} + (\mathbf{v},\mathbf{a})\mathbf{a}^{o}\big)\varepsilon.$$

Substituting into the formula for $R\mathbf{v}R$ we get:

$$\begin{aligned}&\cos^2\frac{\theta}{2}\mathbf{v} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(\mathbf{a}\times\mathbf{v} + \mathbf{a}^{o}\times\mathbf{v}\varepsilon\right) - \sin^2\frac{\theta}{2}\left(\mathbf{v} - 2\mathbf{a}(\mathbf{v},\mathbf{a}) - 2\big((\mathbf{v},\mathbf{a}^{o})\mathbf{a} + (\mathbf{v},\mathbf{a})\mathbf{a}^{o}\big)\varepsilon\right) = \\ &= \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\mathbf{v} + \sin\theta\left(\mathbf{a}\times\mathbf{v} + \mathbf{a}^{o}\times\mathbf{v}\varepsilon\right) + 2\sin^2\frac{\theta}{2}\left(\mathbf{a}(\mathbf{v},\mathbf{a}) + \big((\mathbf{v},\mathbf{a}^{o})\mathbf{a} + (\mathbf{v},\mathbf{a})\mathbf{a}^{o}\big)\varepsilon\right) = \\ &= \cos\theta\mathbf{v} + \sin\theta\mathbf{a}\times\mathbf{v} + (1-\cos\theta)\mathbf{a}(\mathbf{v},\mathbf{a}) + \left[\sin\theta\mathbf{a}^{o}\times\mathbf{v} + (1-\cos\theta)\big((\mathbf{v},\mathbf{a}^{o})\mathbf{a} + (\mathbf{v},\mathbf{a})\mathbf{a}^{o}\big)\right]\varepsilon.\end{aligned}$$

The expression $R\mathbf{m}R^{*}\varepsilon$ is calculated in a completely similar way and is significantly simplified by multiplying by the dual imaginary unit ε:

$$R\mathbf{m}R^{*}\varepsilon = \left(\cos\theta\mathbf{m} + \sin\theta\mathbf{a}\times\mathbf{m} + (1-\cos\theta)\mathbf{a}(\mathbf{m},\mathbf{a})\right)\varepsilon.$$

Now we can write a sandwich formula for the rotation of a line around an axis:

$$\mathbf{L}' = R\mathbf{L}R^{*} = \mathbf{v}' + \mathbf{m}'\varepsilon,$$

where:

$$\mathbf{v}' = \cos\theta\mathbf{v} + \sin\theta\mathbf{a}\times\mathbf{v} + (1-\cos\theta)\mathbf{a}(\mathbf{v},\mathbf{a}),$$

$$\mathbf{m}' = \cos\theta\mathbf{m} + \sin\theta\mathbf{a}\times\mathbf{m} + (1-\cos\theta)\mathbf{a}(\mathbf{m},\mathbf{a}) + \sin\theta\mathbf{a}^{o}\times\mathbf{v} + (1-\cos\theta)\big((\mathbf{v},\mathbf{a}^{o})\mathbf{a} + (\mathbf{v},\mathbf{a})\mathbf{a}^{o}\big).$$

To check this equality, we can take Rodrigues' formula and use the principle of transfer to write:

$$\mathbf{L}' = \cos\theta\mathbf{L} + \sin\theta\mathbf{A}\times\mathbf{L} + (1-\cos\theta)(\mathbf{A},\mathbf{L})\mathbf{A},$$

where

$$\mathbf{L} = \mathbf{v} + \mathbf{m}\varepsilon,$$

$$\mathbf{A} = \mathbf{a} + \mathbf{a}^o\varepsilon.$$

The arguments of the functions cos and sin are real, since we have pure rotation without translation:

$$\mathbf{A} \times \mathbf{L} = \mathbf{a} \times \mathbf{v} + (\mathbf{a} \times \mathbf{m} + \mathbf{a}^o \times \mathbf{v})\varepsilon,$$

$$(\mathbf{A}, \mathbf{L})\mathbf{A} = (\mathbf{a}, \mathbf{v})\mathbf{a} + [(\mathbf{a}, \mathbf{m})\mathbf{a} + (\mathbf{a}^o, \mathbf{v})\mathbf{a}]\varepsilon + (\mathbf{a}, \mathbf{v})\mathbf{a}^o\varepsilon.$$

Combining the expressions we obtain:

$$\begin{aligned}\mathbf{L}' = {} & \cos\theta\mathbf{v} + \sin\theta\mathbf{a} \times \mathbf{v} + (1 - \cos\theta)\mathbf{a}(\mathbf{v}, \mathbf{a}) + [\cos\theta\mathbf{m} + \sin\theta\mathbf{a} \times \mathbf{m} + (1 - \cos\theta)(\mathbf{a}, \mathbf{m})\mathbf{a}]\,\varepsilon + \\ & + [\sin\theta\mathbf{a}^o \times \mathbf{v} + (1 - \cos\theta)\,((\mathbf{a}^o, \mathbf{v})\mathbf{a} + (\mathbf{a}, \mathbf{v})\mathbf{a}^o)]\,\varepsilon = \\ & = R_{\theta,\mathbf{a}}(\mathbf{v}) + R_{\theta,\mathbf{a}}(\mathbf{m})\varepsilon + \big[\sin\theta\mathbf{a}^o \times \mathbf{v} + (1 - \cos\theta)\big((\mathbf{v}, \mathbf{a}^o)\mathbf{a} + (\mathbf{v}, \mathbf{a})\mathbf{a}^o\big)\big]\,\varepsilon,\end{aligned}$$

which exactly repeats the result obtained from the dual quaternion formula.

2.5. Screw motion of a plane

Let us recall that the dual quaternion representation of a plane has the following form:

$$\Pi = ai + bj + ck + d\varepsilon = \mathbf{n} + d\varepsilon.$$

where $\mathbf{n}$ is the unit direction normal vector, and d is the distance from the origin to the plane. The sandwich formula for the screw motion of the plane will have the form [13, pp. 49–50]:

$$\Pi' = \Lambda\Pi\Lambda^\dagger,$$

where the dual quaternion of screw motion is already known to us:

$$\Lambda = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon) + \left(\cos\frac{\theta}{2}\mathbf{a} - \sin\frac{\theta}{2}\right)\frac{\theta^o}{2}\varepsilon.$$

2.5.1. Plane Translation

Consider the translation $T = 1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon$. Note that:

$$TT = \left(1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon\right)^2 = 1 + 2\frac{\theta^o}{2}\mathbf{a}\varepsilon = 1 + \theta^o\mathbf{a}\varepsilon,$$

$$T\Pi T^\dagger = T\Pi T = Td\varepsilon T + T\mathbf{n}T.$$

Let's carry out the calculations separately:

$$Td\varepsilon T = d\varepsilon TT = d\varepsilon + 2\theta^o\mathbf{a}\varepsilon d\varepsilon = d\varepsilon,$$

$$\begin{aligned}T\mathbf{n}T = {} & \left(1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon\right)\mathbf{n}\left(1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon\right) = \left(1 + \frac{\theta^o}{2}\mathbf{a}\varepsilon\right)\left(\mathbf{n} + \frac{\theta^o}{2}\mathbf{n}\mathbf{a}\varepsilon\right) = \\ & = \mathbf{n} + \frac{\theta^o}{2}\mathbf{n}\mathbf{a}\varepsilon + \frac{\theta^o}{2}\mathbf{a}\mathbf{n}\varepsilon = \mathbf{n} + \frac{\theta^o}{2}(\mathbf{n}\mathbf{a} + \mathbf{a}\mathbf{n})\varepsilon = \mathbf{n} - \theta^o(\mathbf{a}, \mathbf{n})\varepsilon\end{aligned}$$

because the product of vectors:

$$\mathbf{n}\mathbf{a} + \mathbf{a}\mathbf{n} = -(\mathbf{n}, \mathbf{a}) + \mathbf{n} \times \mathbf{a} - (\mathbf{a}, \mathbf{n}) + \mathbf{a} \times \mathbf{n} = -2(\mathbf{a}, \mathbf{n}).$$

As a result we get:

$$\Pi' = T\Pi T = \mathbf{n} + (d - \theta^o(\mathbf{a}, \mathbf{n}))\varepsilon$$

2.5.2. Rotation of a plane

Consider now the pure rotation $R = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon)$ and $R^\dagger = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon)$. Using the formula (9), we write:

$$R(\mathbf{n} + d\varepsilon)R^\dagger = R\mathbf{n}R^\dagger + d\varepsilon RR^\dagger = R\mathbf{n}R^\dagger + d\varepsilon,$$

The calculation of $R\mathbf{n}R^\dagger$ is similar to the calculation we already did for a straight line, but it differs from it in details, so we will also present all the calculations in detail:

$$\begin{aligned}
&\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}(\mathbf{a} + \mathbf{a}^o\varepsilon)\right)\left(\cos\frac{\theta}{2}\mathbf{n} - \sin\frac{\theta}{2}(\mathbf{na} - \mathbf{na}^o\varepsilon)\right) = \\
&= \cos^2\frac{\theta}{2}\mathbf{n} - \cos\frac{\theta}{2}\sin\frac{\theta}{2}(\mathbf{na} - \mathbf{na}^o\varepsilon) + \sin\frac{\theta}{2}\cos\frac{\theta}{2}(\mathbf{an} + \mathbf{a}^o\mathbf{n}\varepsilon) - \sin^2\frac{\theta}{2}(\mathbf{ana} - \mathbf{ana}^o\varepsilon + \mathbf{a}^o\mathbf{na}\varepsilon) = \\
&= \cos^2\frac{\theta}{2}\mathbf{n} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\,(\mathbf{a}\times\mathbf{n} - (\mathbf{a}^o, \mathbf{n})\varepsilon) - \sin^2\frac{\theta}{2}\,(\mathbf{n} - 2(\mathbf{n}, \mathbf{a})\mathbf{a} + 2(\mathbf{a}, \mathbf{n}, \mathbf{a}^o)\varepsilon) = \\
&= \cos\theta\mathbf{n} + \sin\theta\,(\mathbf{a}\times\mathbf{n} - (\mathbf{a}^o, \mathbf{n})\varepsilon) + (1 - \cos\theta)\,[(\mathbf{n}, \mathbf{a})\mathbf{a} - (\mathbf{a}, \mathbf{n}, \mathbf{a}^o)\varepsilon] = \\
&= \cos\theta\mathbf{n} + \sin\theta\mathbf{a}\times\mathbf{n} + (1 - \cos\theta)(\mathbf{n}, \mathbf{a})\mathbf{a} - [\sin\theta(\mathbf{a}^o, \mathbf{n}) + (\mathbf{a}, \mathbf{n}, \mathbf{a}^o)(1 - \cos\theta)]\,\varepsilon.
\end{aligned}$$

The following simplifications were used during the calculation:

$$-\mathbf{na} + \mathbf{na}^o\varepsilon + \mathbf{an} + \mathbf{a}^o\mathbf{n}\varepsilon = (\mathbf{an} - \mathbf{na}) + (\mathbf{a}^o\mathbf{n} + \mathbf{na}^o)\varepsilon = 2\mathbf{a}\times\mathbf{n} - 2(\mathbf{a}^o, \mathbf{n})\varepsilon,$$

$$\mathbf{ana} = \|\mathbf{a}\|^2\mathbf{n} - 2(\mathbf{n}, \mathbf{a})\mathbf{a} = \mathbf{n} - 2(\mathbf{n}, \mathbf{a})\mathbf{a},$$

$$\begin{aligned}
\mathbf{a}^o\mathbf{na} &= -\mathbf{a}^o(\mathbf{n}, \mathbf{a}) + \mathbf{a}^o\mathbf{n}\times\mathbf{a} = -\mathbf{a}^o(\mathbf{n}, \mathbf{a}) - (\mathbf{a}^o, \mathbf{n}\times\mathbf{a}) + \mathbf{a}^o\times\mathbf{n}\times\mathbf{a} = \\
&= -\mathbf{a}^o(\mathbf{n}, \mathbf{a}) - (\mathbf{a}^o, \mathbf{n}\times\mathbf{a}) + \mathbf{n}(\mathbf{a}^o, \mathbf{a}) - \mathbf{a}(\mathbf{a}^o, \mathbf{n}),
\end{aligned}$$

$$\begin{aligned}
\mathbf{ana}^o &= -\mathbf{a}(\mathbf{n}, \mathbf{a}^o) + \mathbf{an}\times\mathbf{a}^o = -\mathbf{a}(\mathbf{n}, \mathbf{a}^o) - (\mathbf{a}, \mathbf{n}\times\mathbf{a}^o) + \mathbf{a}\times\mathbf{n}\times\mathbf{a}^o = \\
&= -\mathbf{a}(\mathbf{n}, \mathbf{a}^o) - (\mathbf{a}, \mathbf{n}\times\mathbf{a}^o) + \mathbf{n}(\mathbf{a}, \mathbf{a}^o) - \mathbf{a}^o(\mathbf{a}, \mathbf{n}),
\end{aligned}$$

$$\mathbf{a}^o\mathbf{na} - \mathbf{ana}^o = -(\mathbf{a}^o, \mathbf{n}\times\mathbf{a}) - (\mathbf{a}, \mathbf{n}\times\mathbf{a}^o) = +(\mathbf{a}, \mathbf{n}, \mathbf{a}^o) + (\mathbf{a}, \mathbf{n}, \mathbf{a}^o) = 2(\mathbf{a}, \mathbf{n}, \mathbf{a}^o).$$

The result is that:

$$\Pi' = R\Pi R^\dagger = \cos\theta\mathbf{n} + \sin\theta\mathbf{a}\times\mathbf{n} + (1 - \cos\theta)(\mathbf{n}, \mathbf{a})\mathbf{a} + [d - \sin\theta(\mathbf{a}^o, \mathbf{n}) - (1 - \cos\theta)(\mathbf{a}, \mathbf{n}, \mathbf{a}^o)]\,\varepsilon$$

In [13] there is only a formula for rotation around an axis passing through the origin, but there is no formula for an arbitrary axis of rotation.

2.6. Relationship of dual quaternions to projective transformation matrices in $\mathbb{RP}^3$

Let us write the dual quaternion that defines the screw motion in quaternion form:

$$\Lambda = \lambda + \lambda^o\varepsilon = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{a} + \left(\sin\frac{\theta}{2}\mathbf{a}^o + \frac{\theta^o}{2}\cos\frac{\theta}{2}\mathbf{a} - \frac{\theta^o}{2}\sin\frac{\theta}{2}\right)\varepsilon,$$

$$\lambda = \lambda_0 + \lambda_1\mathrm{i} + \lambda_2\mathrm{j} + \lambda_3\mathrm{k} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{a}, \quad \lambda^o = \lambda_0^o + \lambda_1^o\mathrm{i} + \lambda_2^o\mathrm{j} + \lambda_3^o\mathrm{k} = \sin\frac{\theta}{2}\mathbf{a}^o + \frac{\theta^o}{2}\cos\frac{\theta}{2}\mathbf{a} - \frac{\theta^o}{2}\sin\frac{\theta}{2}.$$

The quaternion λ specifies the rotation, and the quaternion λ^o is responsible for translation operations.

Table 2

Relationship of dual quaternions to projective transformation matrices

Comparison criterion	**Matrixes**	**Dual quaternions**
Number of scalar coefficients	$4 \times 4 = 16$	$4 + 4 = 8$
Multiplications (compositions)	48 scalar multiplications	48 scalar multiplications
Point motion	12 scalar multiplications	96 scalar multiplications

The projective transformation matrix that defines rotation and translation looks like this:

$$M = \left[\begin{array}{c|c} R & \mathbf{t} \\ \hline \mathbf{0}^T & 1 \end{array}\right] = \left[\begin{array}{ccc|c} r_1^1 & r_2^1 & r_3^1 & t_x \\ r_1^2 & r_2^2 & r_3^2 & t_y \\ r_1^3 & r_2^3 & r_3^3 & t_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

where the matrix R is written in terms of the quaternion coefficients $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ as

$$R = \begin{bmatrix} \lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2 & 2(\lambda_1\lambda_2 + \lambda_0\lambda_3) & 2(\lambda_1\lambda_3 + \lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2 + \lambda_0\lambda_3) & \lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2 & 2(\lambda_2\lambda_3 - \lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3 - \lambda_0\lambda_2) & 2(\lambda_2\lambda_3 + \lambda_0\lambda_1) & \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2 \end{bmatrix}.$$

Let us show that the column vector **t** can be calculated using the formula

$$\mathbf{t} = 2\lambda^o\lambda^*.$$

Using the Plücker condition $(\mathbf{a}, \mathbf{a}^o) = 0$ and the unity of the vector **a**, we can prove:

$$2\lambda^o\lambda^* = \theta^o\mathbf{a} + \sin\theta\mathbf{a}^o + 2\sin^2\frac{\theta}{2}\mathbf{a} \times \mathbf{a}^o = \theta^o\mathbf{a} + \sin\theta\mathbf{a}^o + (1 - \cos\theta)\mathbf{a} \times \mathbf{a}^o.$$

This formula has two terms:

- $\sin\theta\mathbf{a}^o + (1 - \cos\theta)\mathbf{a} \times \mathbf{a}^o$ — coincides with the part of the formula for rotation around an arbitrary axis (translation and return to the origin);
- $\theta^o\mathbf{a}$ — specifies a translation along the **a** axis by a distance θ.

If the matrix M is given, then the principal part λ of the dual quaternion Λ can be calculated using a special algorithm, described in detail in section 1.4.3, and the calculation of λ^o can be carried out using the formula:

$$\lambda^o = \frac{1}{2}\mathbf{t}\lambda = \frac{1}{2}t\lambda, \quad t = 0 + \mathbf{t}.$$

The formula is valid due to (see table 2):

$$\mathbf{t} = 2\lambda^o\lambda^* \quad \Rightarrow \quad \frac{1}{2}\mathbf{t}\lambda = \frac{1}{2}2\lambda^o\underbrace{\lambda^*\lambda}_{=1} = \lambda^o.$$

- Dual quaternions are computationally less efficient than matrices.
- Dual quaternions are more convenient for defining an arbitrary axis.
- Dual quaternions are easier to use for making heuristic inferences.

2.7. Reflection about a plane using dual quaternions

For a point P represented by a dual quaternion $P = w + \mathbf{p}\varepsilon = w + (x\mathrm{i} + y\mathrm{j} + z\mathrm{k})\varepsilon$, the reflection formula with respect to the plane represented by the dual quaternion $N = \mathbf{n} + d\varepsilon$, where $\|\mathbf{n}\| = 1$, has the following form:

$$P' = -NP^*N.$$

A similar reflection formula is valid for an arbitrary plane π, represented by the dual quaternion $\Pi = \ + \delta\varepsilon$:

$$\Pi' = -N\Pi^*N.$$

In the case of reflection of a straight line represented by the dual quaternion $L = \mathbf{v} + \mathbf{m}\varepsilon$, a different formula is valid:

$$L' = NL^{\dagger}N^{-1}.$$

Since usually $\|\mathbf{n}\| = 1$, then $|N|^2 = \|\mathbf{n}\|^2 = 1$ and $N^{-1} = N^*/|N|^2 = N^*$.

Let us prove the formulas.

2.7.1. Point reflection

$$\begin{aligned} P' = -NP^*N = -(\mathbf{n} + d\varepsilon)(w - \mathbf{p}\varepsilon)(\mathbf{n} + d\varepsilon) = -(\mathbf{n} + d\varepsilon)(w\mathbf{n} + dw\varepsilon - \mathbf{pn}\varepsilon - d\mathbf{p}\varepsilon^2) = \\ = -(\mathbf{n} + d\varepsilon)(w\mathbf{n} + dw\varepsilon - \mathbf{pn}\varepsilon) = -(w\mathbf{nn} + dw\mathbf{n}\varepsilon - \mathbf{npn}\varepsilon + dw\mathbf{n}\varepsilon + d^2w\varepsilon^2 - \mathbf{pn}d\varepsilon^2) = \\ = -(-w + 2dw\mathbf{n}\varepsilon - \mathbf{npn}\varepsilon) = w + \mathbf{npn}\varepsilon - 2dw\mathbf{n}\varepsilon = \\ = w + (\mathbf{npn} - 2dw\mathbf{n})\varepsilon = w + (\mathbf{p} - 2(\mathbf{p}, \mathbf{n})\mathbf{n} - 2dw\mathbf{n})\varepsilon. \end{aligned}$$

It can be seen that in the dual part of the dual quaternion P' we obtain exactly the same expression as in the formula (6), if the point is affine and $w = 1$.

For an affine point the expression is somewhat simplified:

$$P' = -NP^*N = 1 + \Big(\mathbf{p} - 2(\mathbf{p}, \mathbf{n})\mathbf{n} - 2d\mathbf{n}\Big)\varepsilon = 1 + (\mathbf{npn} - 2d\mathbf{n})\varepsilon.$$

2.7.2. Reflection of a plane

The reflected plane is defined by the dual quaternion $\Pi = \ + \delta\varepsilon$, and the plane relative to which the reflection occurs is represented by the dual quaternion $N = \mathbf{n} + d\varepsilon$, $\|\mathbf{n}\| = 1$.

Note that $\Pi^* = -\nu + \delta\varepsilon$ and write

$$\begin{aligned} -N\Pi^*N = -(\mathbf{n} + d\varepsilon)(- + \delta\varepsilon)(\mathbf{n} + d\varepsilon) = \\ = -(\mathbf{n} + d\varepsilon)(-\mathbf{n} - d\varepsilon + \delta\mathbf{n}\varepsilon + \delta d\varepsilon^2) = -(\mathbf{n} + d\varepsilon)(-\mathbf{n} - d\varepsilon + \delta\mathbf{n}\varepsilon) = \\ = -(-\mathbf{nn} - d\mathbf{n}\varepsilon + \delta\mathbf{nn}\varepsilon - d\mathbf{n}\varepsilon - d^2\varepsilon^2 + \delta d\mathbf{n}\varepsilon^2) = \mathbf{nn} + d\mathbf{n}\varepsilon + \delta\varepsilon + d\mathbf{n}\varepsilon = \\ = \mathbf{nn} + d(\mathbf{n} + \mathbf{n})\varepsilon + \delta\varepsilon = \ - 2(, \mathbf{n})\mathbf{n} - \big(2d(, \mathbf{n}) - \delta\big)\varepsilon. \end{aligned}$$

As a result, we obtain the formula:

$$-N\Pi^*N = \ - 2(, \mathbf{n})\mathbf{n} - (2d(, \mathbf{n}) - \delta)\varepsilon.$$

This formula can be rewritten in a different form if we consider the case when the planes N and Π are not parallel and have a common point P, then

$$\delta\varepsilon = -(P,) = -(1 + \mathbf{p}\varepsilon,) = -(1,) - (\mathbf{p},)\varepsilon = -(\mathbf{p},)\varepsilon, \quad d\varepsilon = -(P, \mathbf{n}).$$

Using these two relations, the plane reflection formula can be written as follows:

$$-N\Pi^{*}N = -2(,\mathbf{n})\mathbf{n} - (-2(P,\mathbf{n})(,\mathbf{n}) + (P,)) = -2(,\mathbf{n})\mathbf{n} - ((P,-2(,\mathbf{n})\mathbf{n}) + (P,)) =$$
$$= -2(,\mathbf{n})\mathbf{n} - (P,\ -2(,\mathbf{n})\mathbf{n}) = -2(,\mathbf{n})\mathbf{n} - (P,\ -2(,\mathbf{n})\mathbf{n}) = \mathbf{nn} - (P,\mathbf{nn}).$$

2.7.3. Reflection of a line

We will assume that the normal vector of the plane relative to which the reflection occurs is unitary. Then $N^{-1} = N^{*}/|N| = N^{*}/\|\mathbf{n}\| = N^{*}$. As a result, $N^{-1} = N^{*} = \mathbf{n}^{*} + d\varepsilon = -\mathbf{n} + d\varepsilon$.

Let us now find the dual conjugation of the dual quaternion of the line L

$$L^{\dagger} = (\mathbf{v} + \mathbf{m}\varepsilon)^{\dagger} = \mathbf{v}^{*} - \mathbf{m}\varepsilon = -\mathbf{v} + \mathbf{m}\varepsilon.$$

Now we transform the expression

$$L' = NL^{\dagger}N^{-1} = (\mathbf{n} + d\varepsilon)(-\mathbf{v} + \mathbf{m}\varepsilon)(-\mathbf{n} + d\varepsilon) = (\mathbf{n} + d\varepsilon)(\mathbf{vn} - d\mathbf{v}\varepsilon - \mathbf{mn}\varepsilon + d\mathbf{m}\varepsilon^{2}) =$$
$$= (\mathbf{n} + d\varepsilon)(\mathbf{vn} - d\mathbf{v}\varepsilon - \mathbf{mn}\varepsilon) = \mathbf{nvn} - d\mathbf{nv}\varepsilon - \mathbf{nmn}\varepsilon + d\mathbf{vn}\varepsilon - d^{2}\mathbf{v}\varepsilon^{2} - d\mathbf{mn}\varepsilon^{2} =$$
$$= \mathbf{nvn} - \mathbf{nmn}\varepsilon + d(\mathbf{vn} - \mathbf{nv})\varepsilon = \mathbf{nvn} - \mathbf{nmn}\varepsilon + 2d\mathbf{v} \times \mathbf{n}\varepsilon =$$
$$= \|\mathbf{n}\|^{2}\mathbf{v} - 2(\mathbf{v},\mathbf{n})\mathbf{n} - (\|\mathbf{n}\|^{2}\mathbf{m} - 2(\mathbf{m},\mathbf{n})\mathbf{n})\varepsilon + 2d\mathbf{v} \times \mathbf{n}\varepsilon = \mathbf{v} - 2(\mathbf{v},\mathbf{n})\mathbf{n} - (\mathbf{m} - 2(\mathbf{m},\mathbf{n})\mathbf{n} - 2d\mathbf{v} \times \mathbf{n})\varepsilon.$$

Let us express the moment $\mathbf{m} = \mathbf{p} \times \mathbf{v}$ through the point P on the line and the vector $\mathbf{v}$. In this case, we take into account that $\mathbf{v} - 2(\mathbf{v},\mathbf{n})\mathbf{n} = \mathbf{nvn}$ and $\mathbf{m} - 2(\mathbf{m},\mathbf{n})\mathbf{n} = \mathbf{nmn}$, then

$$NL^{\dagger}N^{-1} = \mathbf{nvn} - \mathbf{n}(\mathbf{p} \times \mathbf{v})\mathbf{n}\varepsilon + 2d\mathbf{v} \times \mathbf{n}\varepsilon = \mathbf{nvn} + (\mathbf{npn}) \times (\mathbf{nvn})\varepsilon + 2d\mathbf{v} \times \mathbf{n}\varepsilon.$$

Consider the expression $2d\mathbf{v} \times \mathbf{n}$ and transform it as follows:

$$\mathbf{nvn} = \mathbf{v} - 2(\mathbf{v},\mathbf{n})\mathbf{n} \Rightarrow \mathbf{v} = \mathbf{nvn} + 2(\mathbf{v},\mathbf{n})\mathbf{n},$$
$$2d\mathbf{v} \times \mathbf{n} = -2d\mathbf{n} \times \mathbf{v} = -2d\mathbf{n} \times (\mathbf{nvn} + 2(\mathbf{v},\mathbf{n})\mathbf{n}) =$$
$$= -2d\mathbf{n} \times (\mathbf{nvn}) - 2d(\mathbf{v},\mathbf{n})\mathbf{n} \times \mathbf{n} = -2d\mathbf{n} \times (\mathbf{nvn}),$$
$$2d\mathbf{v} \times \mathbf{n} = -2d\mathbf{n} \times (\mathbf{nvn}).$$

Let's substitute into the basic formula for reflection:

$$NL^{\dagger}N^{-1} = \mathbf{nvn} + (\mathbf{npn}) \times (\mathbf{nvn})\varepsilon - 2d\mathbf{n} \times (\mathbf{nvn}) = \mathbf{nvn} + (\mathbf{npn} - 2d\mathbf{n}) \times (\mathbf{nvn})\varepsilon.$$

Above we obtained the formula $P' = -NP^{*}N = 1 + (\mathbf{npn} - 2d\mathbf{n})\varepsilon$, with the help of which we can write

$$(\mathbf{npn} - 2d\mathbf{n})\varepsilon = (P' - 1)$$
$$NL^{\dagger}N^{-1} = \mathbf{nvn} + (P' - 1) \times (\mathbf{nvn}) = \mathbf{nvn} + (-NP^{*}N - 1) \times (\mathbf{nvn}).$$

In this form, this formula is written in [13, p. 59]. As a result, we have the formula in three versions:

$$L' = NL^{\dagger}N^{-1} = \mathbf{nvn} + (2d\mathbf{v} \times \mathbf{n} - \mathbf{nmn})\varepsilon = \mathbf{v} - 2(\mathbf{v},\mathbf{n})\mathbf{n} - (\mathbf{m} - 2(\mathbf{m},\mathbf{n})\mathbf{n} - 2d\mathbf{v} \times \mathbf{n})\varepsilon =$$
$$= \mathbf{nvn} + (-NP^{*}N - 1) \times (\mathbf{nvn}).$$

3. Results

As a result of the work, dual quaternionic formulas were obtained for

- of screw motion of a straight line, point, and plane;
- for pure rotation and pure parallel transfer (translation) of straight lines, points and planes;
- formula for calculating the projective matrix of screw motion according to a given dual quaternion;
- formulas for reflecting points, straight lines, and planes relative to randomly positioned planes.

We dare to attribute to one of the results the very fact of detailed derivation of these formulas, as this will make it easier for readers of the article to master working with dual quaternions.

4. Discussion

The material presented in this article shows that dual quaternions can be used to comprehensively describe both proper and improper movements in three-dimensional space. Currently, there are many alternative formalisms, the most popular and actively promoted of which is the geometric algebra formalism.

We will not go into a comparison of dual quaternions and geometric algebra here, since without a detailed presentation of the latter in terms consistent with dual quaternions, this comparison is unproductive. However, we note the following.

- In our opinion, when applying the Cayley–Dickson procedure, dual quaternions cannot be called some kind of artificial formation and they are quite logical.
- All dual quaternion formulas, when disclosed, are reduced to standard calculations with three-dimensional vectors, which makes it possible to efficiently implement calculations using, for example, shader languages where support for these vectors is implemented at the hardware level.

5. Conclusion

All the formulas obtained are of practical interest, as they can be used to calculate finite complex movements, as well as to construct surfaces (for example, linear ones). In future publications, we plan to use them to visualize various examples of movement, focusing on software implementation. The detailed material in this article will allow you not to be distracted by mathematical details and focus on the subtleties of software implementation and the algorithmic side of the issue.

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Бикватернионное представление движения в трёхмерном пространстве

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Аннотация. *Предпосылки* В предыдущей статье авторов был подробно рассмотрен вопрос использования бикватернионов для задания точек, прямых и плоскостей и решения стандартных геометрических задач. Данная статья является логическим продолжением и раскрывает применение бикватернионов для описания изометрий трёхмерного пространства. *Цель* Вывод всех необходимых формул для винтового движения точек, прямых и плоскостей, а также зеркальной симметрии (отражения) относительно плоскости. Доработка обозначений и формализма. *Методы* Используется алгебра дуальных чисел, кватернионов и бикватернионов, а также элементы теории винтов и скользящих векторов. *Результаты* Получены и систематизированы формулы для вращения, трансляции, отражения, винтового движения и зеркального вращения. *Выводы* Бикватернионы могут служить полноценным инструментом для описания винтового движения в пространстве. Благодаря возможности выражения бикватернионных операций через стандартное векторное и скалярное произведения, полученные формулы допускают эффективную программную реализацию.

Ключевые слова: натурное моделирование, воспроизводимое исследование, исследование как код