Discrete & Continuous Models
& Applied Computational Science
ISSN 2658-7149 (Online), 2658-4670 (Print)

2026, 34 (1) 40–54
http://journals.rudn.ru/miph

Research article

UDC 519.872, 519.217
PACS 07.05.Tp, 02.60.Pn, 02.70.Bf
DOI: 10.22363/2658-4670-2026-34-1-40-54
EDN: VEJQIO

Derivative-free iterations in R^n with point-wise operations for solving systems of nonlinear equations

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(received: May 12, 2025; revised: September 30, 2025; accepted: October 30, 2025)

Abstract. In this paper, we develop a new family of high-order derivative-free iterative methods for solving systems of nonlinear equations. Specifically, we propose four two-step derivative-free schemes with convergence orders four and five, together with twelve three-step derivative-free schemes achieving convergence orders six, seven, and eight. The main specific of these iterations is that they include a vector or even a scalar iteration parameter instead of the matrix parameter inherent to other existing iterative methods. This structural simplification significantly reduces computational cost, storage requirements, and matrix operations, thereby improving overall computational efficiency. A convergence analysis is presented, establishing the theoretical order of convergence of the proposed methods. The efficiency indices of the proposed schemes are derived and compared with those of several well-known derivative-free iterative methods. The numerical experiments on standard academic problems confirm the theoretical results and demonstrate that the proposed methods are competitive and, in many cases, superior in terms of efficiency and robustness.

Key words and phrases: nonlinear systems, derivative-free iterations, efficiency index, order of convergence

For citation: Zhanlav, T., Otgondorj, K., Ulziibayar, V., Enkhbayar, K. Derivative-free iterations in R^n with point-wise operations for solving systems of nonlinear equations. *Discrete and Continuous Models and Applied Computational Science* **34** (1), 40–54. doi: 10.22363/2658-4670-2026-34-1-40-54. edn: VEJQIO (2026).

1. Introduction

We consider the following nonlinear system of equations:

$$F(x) = 0, \quad x = (x_1, x_2, \cdots, x_n)^T \in R^n, \tag{1}$$

where $F : D \subseteq R^n \to R^n$ is a nonlinear and sufficiently Fréchet differentiable function in an open convex set D. Additionally, $F'(x)$ is continuous and nonsingular at α, where α is the simple and isolated solution of equation (1). Most physical systems are inherently nonlinear nature and described by nonlinear systems. The nonlinear systems (1) also appear in many fields of applied sciences and engineering [1–12]. The solution of equation (1) cannot be computed exactly and is often approximated using iterative methods with different orders of convergence. A quite recently have been appeared some papers devoted to the constructing high efficient iterative methods containing vector and even scalar parameter coefficients [1, 3, 4, 6, 13–15]. For obtaining the numerical solution of the system (1) often used the following two-step and three-step iterative methods:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= y_k - \bar{\tau}_k F'(x_k)^{-1}F(y_k), \end{aligned} \tag{2}$$

and

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ z_k &= y_k - \bar{\tau}_k F'(x_k)^{-1}F(y_k), \\ x_{k+1} &= z_k - \alpha_k F'(x_k)^{-1}F(z_k), \end{aligned} \tag{3}$$

where $\bar{\tau}_k$ and α_k are iteration parameters to be determined properly.

The aim of this work is to develop derivative-free version of the iterations (2) and (3) with vector and scalar coefficients. In Section 2, we introduce new derivative-free two-step iterations of orders four and five. In Section 3, we present new derivative-free three-step iterations of order ρ ($\rho = 6, 7, 8$) and an analysis of the efficiency of the proposed iterative methods. Section 4 devoted to analysis of efficiency of proposed methods compared with other methods. In Section 5, we present the results of our experiments and compare them with known methods of the same order. The article concludes with some conclusions and references used in it.

2. The construction of two-step derivative-free iterations

First, we employ R^n with point-wise multiplication and division of vectors. Let $a = (a_1, a_2, \dots, a_n)^T \in R^n$ and $b = (b_1, b_2, \dots, b_n)^T \in R^n$. The point-wise multiplication and division of two vectors are defined by

$$a \cdot b = (a_1 b_1, a_2 b_2, \dots, a_n b_n)^T \in R^n, \tag{4a}$$

$$\frac{a}{b} = \left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}\right)^T \in R^n. \tag{4b}$$

The direct consequence of (4a) and (4b) is

$$\begin{aligned} a^2 &:= (a \cdot a) = (a_1^2, a_2^2, \dots, a_n^2)^T \in R^n, \\ \mathbf{1} &= \left(1, 1, \dots, 1\right)^T \in R^n. \end{aligned}$$

In [6] the following theorems were proven:

Theorem 6. *[6] The two-step iteration* (2) *has a third, fourth and fifth-order convergence if and only if the parameter* $\bar{\tau}_k$ *satisfies*

$$\bar{\tau}_k = \mathbf{1} + O(h),$$

$$\bar{\tau}_k = \mathbf{1} + 2\Theta_k + O(h^2), \tag{6a}$$
$$\bar{\tau}_k F'(x_k)^{-1}F(y_k) = (\mathbf{1} + \Theta_k^2)F'(y_k)^{-1}F(y_k) + O(h^3),$$

Theorem 7. *[6] The three-step iteration* (3) *have order of convergence* $\rho+1$, $\rho+2$, $\rho+3$ *if and only if the parameter* α_k *satisfies*

$$\alpha_k = \mathbf{1} + O(h),$$
$$\alpha_k = \mathbf{1} + 2\Theta_k + O(h^2), \tag{7a}$$
$$\alpha_k F'(x_k)^{-1}F(z_k) = (\mathbf{1} + 2\Theta_k^2)F'(y_k)^{-1}F(z_k) + O(h^3),$$

where

$$\Theta_k = \frac{F(y_k)}{F(x_k)}, \quad \Theta_k^2 = (\Theta_k \cdot \Theta_k),$$

and ρ *is the order of convergence of iteration* (2).

We note that the conditions (6a) and (7a) can be replaced by

$$\bar{\tau}_k = \alpha_k = \frac{\mathbf{1} + a\Theta_k + b\Theta_k^2}{\mathbf{1} + (a-2)\Theta_k + d\Theta_k^2}, \quad a, b, d \in R,$$

and in this case the convergence order maintained. We now proceed with the construction of a derivative-free analog of (2) as follows:

$$y_k = x_k - [w_k, s_k; F]^{-1} F(x_k),$$
$$x_{k+1} = y_k - T_k [w_k, s_k; F]^{-1} F(y_k), \tag{8}$$

where $[w_k, s_k; F]$ is first order divided difference with

$$w_k = x_k + \gamma_1 F(x_k), \quad s_k = x_k - \gamma_1 F(x_k), \quad \gamma_1 \neq 0, \quad \gamma_1 \in R.$$

It is easy to show that

$$T_k = \bar{\tau}_k F'(x_k)^{-1} [w_k, s_k; F], \tag{9}$$

or

$$\bar{\tau}_k = T_k [w_k, s_k; F]^{-1} F'(x_k). \tag{10}$$

The passing of (2) to (8) is realized by (9). The converse is realized by (10). It is easy to show that

$$F'(x_k)^{-1} [w_k, s_k; F] = I + B_k + O(h^4), \tag{11}$$

where

$$B_k = \frac{1}{6} F'(x_k)^{-1} F'''(x_k) \gamma_1^2 F(x_k)^2 = O(h^2). \tag{12}$$

If we take (12) into account, then from (11) it follows that

$$F'(x_k)^{-1} = [w_k, s_k; F]^{-1} + O(h^2). \tag{13}$$

Analogously, using (11) and the Taylor expansion of $F(y_k)$ at point x_k, we easily obtain

$$F(y_k) = O(h^2), \quad F'(y_k) = [u_k, \varpi_k; F] + O(h^4), \tag{14}$$

where

$$u_k = y_k + \beta_1 F(x_k), \quad \varpi_k = y_k - \beta_1 F(x_k), \quad \beta_1 \neq 0, \quad \beta_1 \in R.$$

Using (9), (11), (13) and (14) it is easy to show that the ρ-order conditions (6) can be rewritten in term of T_k as:

$$T_k = \mathbf{1} + O(h),$$

$$T_k = \mathbf{1} + 2\Theta_k + O(h^2) = \frac{\mathbf{1} + 2\Theta_k + b\Theta_k^2}{1 + d\Theta_k^2} + O(h^2), \tag{15a}$$

$$T_k \left[w_k, s_k; F\right]^{-1} F(y_k) = (\mathbf{1} + \Theta_k^2) \left[u_k, \varpi_k; F\right]^{-1} F(y_k). \tag{15b}$$

Using (15a) in (8) we obtain the following family of fourth order iterations (M_1^4)

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ x_{k+1} &= y_k - \frac{1}{\mathbf{1} + d\Theta_k^2} [w_k, s_k; F]^{-1} \left[(\mathbf{1} + b\Theta_k^2) F(y_k) + 2\Theta_k^2 F(x_k)\right], \; d, b \in R. \end{aligned} \tag{16}$$

Analogously, using (15b) in (8) we obtain the following fifth order iteration (M_2^5)

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ x_{k+1} &= y_k - (\mathbf{1} + \Theta_k^2) [u_k, \varpi_k; F]^{-1} F(y_k). \end{aligned} \tag{17}$$

If $\gamma_1 \to 0$ and $\beta_1 \to 0$ then (16) and (17) lead to the iteration with derivative, considered in [6] and in [4]. The scalar coefficients versions of (16) and (17) are [1]

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ x_{k+1} &= y_k - \frac{1}{1 + dv_k} [w_k, s_k; F]^{-1} \left[(1 + bv_k) F(y_k) + 2v_k F(x_k)\right], \; d, b \in R, \end{aligned} \tag{18}$$

$$v_k = \frac{\|F(y_k)\|^2}{\|F(x_k)\|^2},$$

and

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ x_{k+1} &= y_k - (1 + v_k) [u_k, \varpi_k; F]^{-1} F(y_k), \end{aligned} \tag{19}$$

with convergence order 4 and 5 respectively. The iteration (18) completely coincides with scheme given in [13], while (19) can be considered as new scheme with fifth order of convergence. Let's denote the methods (18) and (19) as (M_3^4) and (M_4^5), respectively.

To analyze the convergence behavior of the proposed method, we first present a lemma that will be used to develop the Taylor expansion of vector functions (see [16]).

Lemma 1. *Let $F : D \subseteq R^n \to R^n$ be p-times Fréchet differentiable in a open convex set $D \subseteq R^n$, then for any $x, \hat{h} \in D$ the following expression holds:*

$$F(x + \hat{h}) = F(x) + F'(x)\hat{h} + \frac{1}{2!} F''(x)\hat{h}^2 + \cdots + \frac{1}{(p-1)!} F^{(p-1)}(x)\hat{h}^{p-1} + R_p,$$

where

$$\|R_p\| \leqslant \frac{1}{p!} \sup_{0<t<1} \|F^{(p)}(x + t\hat{h})\| \|\hat{h}\|^p \;\; \text{and} \;\; \hat{h}^p = (\overbrace{\hat{h}, \hat{h}, \cdots, \hat{h}}^{p}),$$

and $\|\cdot\|$ denotes any norm in R^n, or a corresponding operator norm.

Definition 1. *Let $e_k = x_k - \alpha$ be the error in the k-th iteration, we call the relation*

$$e_{k+1} = L(e_k)^p + O((e_k)^{p+1}),$$

as the error equation. Here, p is the order of convergence, L is a p-linear function, i.e. $L \in \mathcal{L}(\overbrace{R^n \times \cdots \times R^n}^{p}, R^n)$.

In the following result, we establish the convergence of the family of methods given by (16) under the conditions stated in Lemma 1.

Theorem 8. *Let the function $F : D \subseteq R^n \to R^n$ be sufficiently differentiable in a convex set D containing a zero α of $F(x)$. Further, assume that $F'(x)$ is continuous and non-singular at α and the initial guess x_0 is sufficiently close to the solution. Then, the sequence generated by method* (16) *converges to the solution α with order four, for any nonzero value of parameter γ_1 and for any values of b and d.*

Proof. By applying the Taylor expansion of $F(x_k)$ around α, we obtain

$$F(x_k) = F'(\alpha)(e_k + A_2e_k^2 + A_3e_k^3 + A_4e_k^4) + O(e_k^5), \tag{20}$$

$$\begin{aligned} F'(x_k) &= F'(\alpha)(I + 2A_2e_k + 3A_3e_k^2 + 4A_4e_k^3) + O(e_k^4), \\ F''(x_k) &= F'(\alpha)(2A_2 + 6A_3e_k + 12A_4e_k^2) + O(e_k^3), \\ F'''(x_k) &= F'(\alpha)(6A_3 + 24A_4e_k) + O(e_k^2), \end{aligned} \tag{21}$$

where

$$A_i = \frac{1}{i!}[F'(\alpha)]^{-1}F^{(i)}(\alpha), \quad i = 2, 3, \dots .$$

Using the Genocchi–Hermite formula [17] and (20)–(21), we obtain

$$\begin{aligned} [w_k, s_k; F] &= F'(x_k) + \frac{1}{6}F'''(x_k)(\gamma_1F(x_k))^2 + O((\gamma_1F(x_k))^3) = \\ &= F'(\alpha)(I + 2A_2e_k + 3A_3e_k^2) + \frac{1}{6}F'(\alpha)6A_3\gamma_1^2F'(\alpha)^2(e_k)^2 + O(e_k^3) = \\ &= F'(\alpha)(I + 2A_2e_k + A_3(3I + \gamma_1^2F'(\alpha)^2)e_k^2) + O(e_k^3). \end{aligned}$$

Inversion of $[w_k, s_k; F]$ yields

$$[w_k, s_k; F]^{-1} = (I + C_1e_k + C_2e_k^2)F'(\alpha)^{-1} + O(e_k^3), \tag{22}$$

where $C_1 = -2A_2$, $C_2 = 4A_2^2 - A_3(3I + \gamma_1^2F'(\alpha)^2)$.

Let us denote $\bar{e}_k = y_k - \alpha$. From (20) and (22), we get

$$\bar{e}_k = x_k - \alpha - [w_k, s_k; F]^{-1}F(x_k) = B_1e_k^2 + B_2e_k^3 + O(e_k^4),$$

where $B_1 = A_2$, $B_2 = -2A_2^2 + A_3(2I + \gamma_1^2F'(\alpha)^2)$. We then obtain

$$F(y_k) = F'(\alpha)(\bar{e}_k + A_2\bar{e}_k^2 + A_3\bar{e}_k^3) + O(e_k^4) = F'(\alpha)(B_1e_k^2 + B_2e_k^3) + O(e_k^4). \tag{23}$$

Next, we expand the term $\Theta_k = \dfrac{F(y_k)}{F(x_k)}$, which appears in the second step of (16). From (20) and (23), we obtain

$$\Theta_k^2 = A_2^2e_k^2 + (-6A_2^3 + 2A_2A_3(2I + \gamma_1^2F'(\alpha)^2))e_k^3 + O(e_k^4). \tag{24}$$

Then, from (24), we can get

$$P_k = I\frac{1+b\Theta_k^2}{1+d\Theta_k^2} = I + A_2^2(b-d)e_k^2 - 2(A_2(b-d)(3A_2^2 - A_3(2I+\gamma_1^2F'(\alpha)^2)))e_k^3 + O(e_k^4), \tag{25}$$

and

$$Q_k = \frac{2\Theta_k^2}{1+d\Theta_k^2} = 2A_2^2e_k^2 + 4A_2(-3A_2^2 + A_3(2I+\gamma_1^2F'(\alpha)^2))e_k^3 + O(e_k^4). \tag{26}$$

From (25) and (26), it follows that

$$\begin{aligned} P_kF(y_k) + Q_kF(x_k) &= A_2^2e_k^2 + A_3(2I+\gamma_1^2F'(\alpha)^2))e_k^3 + \\ &+ A_2^3(-10+b-d) + 4A_2A_3(2I+\gamma_1^2F'(\alpha)^2)e_k^4 + O(e_k^5). \end{aligned} \tag{27}$$

Then, using (22), and (27), the second step of the method (16) gives the error equation as

$$\begin{aligned} e_{k+1} = x_{k+1} - \alpha &= 10A_2^3 - bA_2^3 - 8A_2A_3 + A_2^3d - 4A_2A_3\gamma_1^2F'(\alpha)^2 + \\ &+ 2A_2A_3(2I+\gamma_1^2F'(\alpha)^2) - A_2(4A_2^2 - A_3(3I+\gamma_1^2F'(\alpha)^2))e_k^4 + O(e_k^5) = \\ &= -A_2(A_3 + A_2^2(-6+b-d) + A_3\gamma_1^2F'(\alpha)^2)e_k^4 + O(e_k^5). \end{aligned}$$

This shows the fourth order convergence of the proposed family (16). □

The convergence analysis of the other proposed methods follows a similar approach to the proof of Theorem 8. Therefore, we omit it here.

3. The construction of three-step derivative-free iterations

The derivative-free analogy of iteration (3) obtained as:

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ z_k &= y_k - T_k [w_k, s_k; F]^{-1} F(y_k), \\ x_{k+1} &= z_k - H_k [w_k, s_k; F]^{-1} F(z_k), \end{aligned}$$

where T_k is given by (15) and H_k determined as:

$$H_k = \alpha_k F(x_k)^{-1} [w_k, s_k; F].$$

As before, the condition (7) can be rewritten in term of H_k as:

$$\begin{aligned} H_k &= \mathbf{1} + O(h), \\ H_k &= \mathbf{1} + 2\Theta_k + O(h^2) = \frac{\mathbf{1} + 2\Theta_k + b\Theta_k^2}{1+d\Theta_k^2} + O(h^2),\ b, d \in R, \\ H_k [w_k, s_k; F]^{-1} F(z_k) &= (\mathbf{1} + 2\Theta_k^2) [u_k, \varpi_k; F]^{-1} F(z_k) + O(h^3). \end{aligned}$$

Theorem 7 and the combination of choices (15) and (28) yields different derivative-free three-step iterations and we list some of these methods below.

Sixth-order iterations:

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ z_k &= y_k - \frac{1}{\mathbf{1} + d\Theta_k^2} [w_k, s_k; F]^{-1} \left[(\mathbf{1} + b\Theta_k)F(y_k) + 2\Theta_k^2 F(x_k)\right], \\ x_{k+1} &= z_k - (\mathbf{1} + 2\Theta_k) [w_k, s_k; F]^{-1} F(z_k), \end{aligned} \tag{29}$$

and

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ z_k &= y_k - (\mathbf{1} + \Theta_k^2) [u_k, \varpi_k; F]^{-1} F(y_k), \\ x_{k+1} &= z_k - [w_k, s_k; F]^{-1} F(z_k), \end{aligned}$$

and

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ z_k &= y_k - [w_k, s_k; F]^{-1} F(y_k), \\ x_{k+1} &= z_k - (\mathbf{1} + 2\Theta_k^2) [u_k, \varpi_k; F]^{-1} F(z_k). \end{aligned}$$

Seventh-order iterations

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ z_k &= y_k - \frac{1}{\mathbf{1} + d\Theta_k^2} [w_k, s_k; F]^{-1} \left[(\mathbf{1} + b\Theta_k)F(y_k) + 2\Theta_k^2 F(x_k)\right], \\ x_{k+1} &= z_k - (\mathbf{1} + 2\Theta_k^2) [u_k, \varpi_k; F]^{-1} F(z_k), \end{aligned}$$

and

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ z_k &= y_k - (\mathbf{1} + \Theta_k^2) [u_k, \varpi_k; F]^{-1} F(y_k), \\ x_{k+1} &= z_k - (\mathbf{1} + 2\Theta_k) [w_k, s_k; F]^{-1} F(z_k). \end{aligned}$$

Eighth-order iterations

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ z_k &= y_k - (\mathbf{1} + \Theta_k^2) [u_k, \varpi_k; F]^{-1} F(y_k), \\ x_{k+1} &= z_k - (\mathbf{1} + 2\Theta_k^2) [u_k, \varpi_k; F]^{-1} F(z_k). \end{aligned} \tag{30}$$

In the remainder of the paper, the methods (29)–(30) will be denoted by M_5^6, M_6^6, M_7^6, M_8^7, M_9^7 and M_{10}^8, respectively. If we take the transition rule that established in [14] into account, then we easily obtain from (29)–(30) its scalar coefficients variants:
Sixth-order iterations

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ z_k &= y_k - \frac{1}{1 + dv_k} [w_k, s_k; F]^{-1} \left[(1 + bv_k)F(y_k) + 2v_k F(x_k)\right], \\ x_{k+1} &= z_k - [u_k, \varpi_k; F]^{-1} F(z_k), \end{aligned} \tag{31}$$

where

$$v_k = \frac{\|F(y_k)\|^2}{\|F(x_k)\|^2},$$

and

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ z_k &= y_k - (1 + v_k)\, [u_k, \varpi_k; F]^{-1} F(y_k), \\ x_{k+1} &= z_k - [w_k, s_k; F]^{-1} F(z_k). \end{aligned}$$

and

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ z_k &= y_k - [w_k, s_k; F]^{-1} F(y_k), \\ x_{k+1} &= z_k - (1 + 2v_k)\, [u_k, \varpi_k; F]^{-1} F(z_k). \end{aligned}$$

Seventh-order methods:

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ z_k &= y_k - \frac{1}{1 + dv_k} [w_k, s_k; F]^{-1} \left[(1 + bv_k)F(y_k) + 2v_k F(x_k)\right], \\ x_{k+1} &= z_k - (1 + 2v_k)\, [u_k, \varpi_k; F]^{-1} F(z_k), \end{aligned}$$

and

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ z_k &= y_k - (1 + v_k)\, [u_k, \varpi_k; F]^{-1} F(y_k), \\ x_{k+1} &= z_k - [u_k, \varpi_k; F]^{-1} (F(z_k) - \beta_k F(x_k)), \quad \beta_k = \frac{\|F(z_k)\|^2}{\|F(y_k)\|^2}. \end{aligned}$$

Eight-order method:

$$\begin{aligned} y_k &= x_k - [w_k, s_k; F]^{-1} F(x_k), \\ z_k &= y_k - (1 + v_k)\, [u_k, \varpi_k; F]^{-1} F(y_k), \\ x_{k+1} &= z_k - (1 + 2v_k)\, [u_k, \varpi_k; F]^{-1} F(z_k). \end{aligned} \tag{32}$$

In the rest of the paper, the methods (31)–(32) will be denoted by M^6_{11}, M^6_{12}, M^6_{13}, M^7_{14}, M^7_{15} and M^8_{16}, respectively.

4. Computational efficiency

The computational efficiency index of an iterative method for solving a nonlinear system is defined by $CI = \rho^{\frac{1}{C}}$, where ρ is the order of convergence and C is the computational cost of each method. We will study the computational efficiency of the presented methods and compare it with that of other methods presented in the literature, namely M^4_3 [13], $M_{6,2}$ [18], NM7 [19] and $PM1$ [20]. To compute F in any iterative method we evaluate n scalar functions, whereas the number of scalar evaluations is $n(n-1)$ scalar functions for any divided difference $[\cdot, \cdot; F]$. In addition, we must include the number of operations shown in Table 1.

As we can see in Figs. 1, 2 and in Table 2, in terms of computational efficiency the proposed method M^6_5 is significantly superior to other considered methods. Additionally, fourth-order M^4_3 and eighth-order M^8_{16} also have high computational efficiency.

Table 1

Computational cost of different operations

	Computational cost
LU decomposition	$\frac{1}{3}(n^3 - n)$
Solution of two triangular systems	n^2
Quotients in divided difference operator	n^2
Matrix-vector multiplication	n^2
Scalar-vector multiplication	n
Component-wise multiplication (division) of vectors	n

Table 2

Comparison of computational efficiency

№	methods	ρ	C_i	CI
1	M_5^6	6	$C_1 = \frac{1}{3}n^3 + 5n^2 + \frac{44}{3}n$	$6^{1/C_1}$
2	M_6^6	6	$C_2 = \frac{2}{3}n^3 + 5n^2 + \frac{23}{3}n$	$6^{1/C_2}$
3	M_7^6	6	$C_3 = \frac{2}{3}n^3 + 6n^2 + \frac{28}{3}n$	$6^{1/C_3}$
4	M_8^7	7	$C_4 = \frac{2}{3}n^3 + 6n^2 + \frac{46}{3}n$	$7^{1/C_4}$
5	M_9^7	7	$C_5 = \frac{2}{3}n^3 + 6n^2 + \frac{31}{3}n$	$7^{1/C_5}$
6	M_{10}^8	8	$C_6 = \frac{2}{3}n^3 + 6n^2 + \frac{31}{3}n$	$8^{1/C_6}$
7	M_{11}^6	6	$C_7 = \frac{2}{3}n^3 + 6n^2 + \frac{28}{3}n$	$6^{1/C_7}$
8	M_{12}^6	6	$C_8 = \frac{2}{3}n^3 + 6n^2 + \frac{22}{3}n$	$6^{1/C_8}$
9	M_{13}^6	6	$C_9 = \frac{2}{3}n^3 + 6n^2 + \frac{25}{3}n$	$6^{1/C_9}$
10	M_{14}^7	7	$C_{10} = \frac{2}{3}n^3 + 6n^2 + \frac{31}{3}n$	$7^{1/C_{10}}$
11	M_{15}^7	7	$C_{11} = \frac{2}{3}n^3 + 6n^2 + \frac{28}{3}n$	$7^{1/C_{11}}$
12	M_{16}^8	8	$C_{12} = \frac{2}{3}n^3 + 6n^2 + \frac{25}{3}n$	$8^{1/C_{12}}$

5. Numerical results and discussion

To evaluate the effectiveness of the new method and provide a comparison with existing methods, numerical experiments have been conducted and the results are presented in this section. To achieve this goal, we consider the following nonlinear problems, most of which are the same as in [13, 19, 21].

Example 1. Considering the following system of 20 equations:

$$y_i - \cos\left(2y_i - \sum_{j=1}^{20} y_j\right) = 0, \quad i = 1, 2, \dots, 20.$$

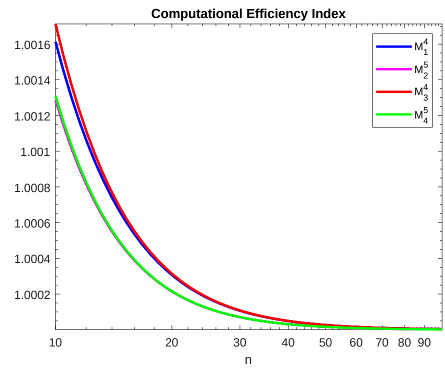

Figure 1. Computational Efficiency Index for $n = 10$ to 100 (logarithmic scale)

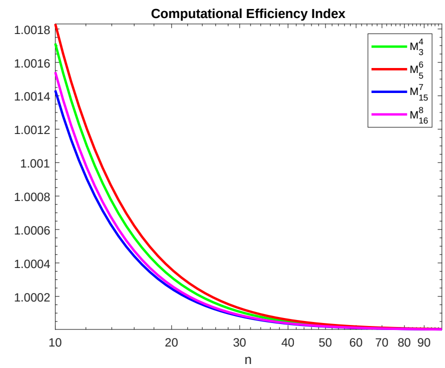

Figure 2. Computational Efficiency Index for $n = 10$ to 100 (logarithmic scale)

The solution is $y^* = \{-0.89, -0.89, \dots, -0.89\}^T$. For this solution, we choose the starting vector $x_0 = \{-0.9, -0.9, \dots, -0.9\}^T$.

Example 2. Consider the system of twenty equations

$$-y_i - 3 + \sum_{j=1}^{20} y_j - e^{y_i} + 4\cos(2\ln(|1 + y_i|)) = 0, \quad i = 1, 2, \dots, 20.$$

The exact solution $y^* = \{0, 0, \dots, 0\}^T$ of above system. For this solution, we choose the starting vector $x_0 = \{0.01, 0.01, \dots, 0.01\}^T$.

In Tables 3 and 4, we present the residual error of the example function $\|F(x_{k+1})\|$, the error between two consecutive iterations $\|x_{k+1} - x_k\|$ and the computational order of convergence ρ_{co}. The computational order of convergence (p_{co}) is calculated using the formula [4]

$$\rho_{co} = \frac{\ln(\|x_{k+1} - x_k\| / \|x_k - x_{k-1}\|)}{\ln(\|x_k - x_{k-1}\| / \|x_{k-1} - x_{k-2}\|)}.$$

The following stopping criterion is used in these experiments:

$$\|x_{k+1} - x_k\| + \|F(x_k)\| \leqslant 10^{-60}.$$

Tables 3 and 4 report the numerical performance of the considered derivative-free iterative methods. The first column lists the names of the methods under comparison. The second column shows the total CPU time (in seconds) required by each method to reach the prescribed stopping criterion. The third column indicates the number of iterations (Iter) needed for convergence.

The fourth column presents the absolute error measured by the norm $\|x_{k+1} - x_k\|$, while the fifth column reports the residual norm $\|F(x_{k+1})\|$ at the final iteration, which reflects the accuracy of the computed solution. The last column displays the approximate computational order of convergence (ACOC), confirming the theoretical convergence order of each method.

Table 3

Comparison numerical results on Example 1

Methods	CPUTime	Iter	$\|x_{k+1} - x_k\|$	$\|F(x_{k+1})\|$	ACOC
M_1^4	0.356	4	1.6016×10^{-76}	4.4188×10^{-300}	4.00
M_3^4	0.344	4	1.6016×10^{-76}	4.4188×10^{-300}	4.00
M_2^5	0.344	4	5.8123×10^{-166}	1.9572×10^{-822}	5.00
M_4^5	0.625	4	5.8123×10^{-166}	1.9572×10^{-822}	5.00
M_5^6	0.343	4	3.4780×10^{-228}	3.2447×10^{-1359}	6.00
M_6^6	0.640	4	9.7496×10^{-287}	1.8481×10^{-1711}	6.00
M_7^6	0.703	4	1.2693×10^{-274}	1.7032×10^{-1638}	6.00
M_{11}^6	0.672	4	7.8812×10^{-283}	4.2801×10^{-1688}	6.00
M_{12}^6	0.735	4	9.7496×10^{-287}	1.8481×10^{-1711}	6.00
M_{13}^6	0.782	4	1.2693×10^{-274}	1.7032×10^{-1638}	6.00
M_8^7	0.656	4	9.7872×10^{-388}	3.0845×10^{-2523}	7.00
M_9^7	0.640	4	2.2881×10^{-402}	2.9211×10^{-2524}	7.00
M_{14}^7	0.734	4	9.7872×10^{-388}	9.3521×10^{-2524}	7.00
M_{15}^7	0.732	4	1.5722×10^{-400}	8.5164×10^{-2524}	7.00
M_{10}^8	0.585	3	1.2259×10^{-79}	2.7310×10^{-624}	8.00
M_{16}^8	0.532	3	1.2259×10^{-79}	2.7310×10^{-624}	8.00
$M_{6,2}$ [18]	1.614	4	1.4692×10^{-107}	6.3590×10^{-633}	6.00
NM7 [19]	2.750	3	3.8545×10^{-71}	1.2384×10^{-487}	7.00
PM1 [20]	1.984	4	2.9442×10^{-271}	1.0079×10^{-2152}	8.00

From Tables 3 and 4, we observe that the M_3^4 iterative method is faster than the considered fourth- and fifth-order methods. Furthermore, Tables 3 and 4 indicate that the proposed M_5^6 method is the fastest among the considered methods with orders $\rho = 6, 7$ and 8. This finding is consistent with the results presented in Section 4. From these tables, it follows that M_{16}^8 is not only faster but also more accurate than the considered seventh- and eighth-order methods. Thus, the eighth-order method M_{16}^8 can be highly useful in practical applications that require high accuracy. In conclusion, the numerical results clearly demonstrate that the proposed derivative-free methods with vector and scalar coefficients are superior to those employing matrix coefficients, both in terms of computational time and overall computational cost.

Conclusions

We obtain family of two-step derivative-free iterations of order 4 and 5 and three-step derivative-free iterations of order 6, 7 and 8 with vector and scalar parameter. The specific of these iterations is that they include vector or even scalar parameter of iteration instead of matrix parameter that inherent to other existing iterative methods. The theoretical conclusions are confirmed by numerical

Table 4

Comparison numerical results on Example 2

Methods	CPUTime	Iter	$\|x_{k+1}-x_k\|$	$\|F(x_{k+1})\|$	ACOC
M_1^4	20.672	4	7.0707×10^{-76}	1.6644×10^{-300}	4.00
M_3^4	20.594	4	7.0707×10^{-76}	1.6644×10^{-300}	4.00
M_2^5	35.572	4	5.9230×10^{-172}	4.1262×10^{-858}	5.00
M_4^5	37.563	4	5.9230×10^{-172}	4.1262×10^{-858}	5.00
M_5^6	20.282	4	2.4149×10^{-219}	1.0347×10^{-1310}	6.00
M_6^6	37.782	4	1.1978×10^{-315}	2.8416×10^{-1893}	6.00
M_7^6	37.532	4	4.4919×10^{-302}	1.5809×10^{-1811}	6.00
M_{11}^6	29.656	3	7.6581×10^{-66}	1.9409×10^{-394}	6.00
M_{12}^6	37.469	4	1.1978×10^{-315}	2.8416×10^{-1893}	6.00
M_{13}^6	37.375	4	4.4919×10^{-302}	1.5809×10^{-1811}	6.00
M_8^7	42.219	4	9.1502×10^{-380}	4.7631×10^{-2654}	7.00
M_9^7	38.344	4	1.7236×10^{-399}	2.0037×10^{-2792}	7.00
M_{14}^7	37.922	4	9.1502×10^{-380}	4.7631×10^{-2654}	7.00
M_{15}^7	37.641	4	2.5776×10^{-400}	3.3521×10^{-2798}	7.00
M_{10}^8	29.906	3	6.0681×10^{-78}	1.3860×10^{-620}	8.00
M_{16}^8	29.391	3	6.0681×10^{-78}	1.3860×10^{-620}	8.00
$M_{6,2}$ [18]	56.801	4	7.8477×10^{-204}	1.4880×10^{-1216}	6.00
NM7 [19]	186.703	3	3.4339×10^{-83}	5.5892×10^{-288}	7.00
PM1 [20]	57.985	3	5.8566×10^{-78}	9.7060×10^{-617}	8.00

experiments. Based on numerical examples, one can conclude that our proposed iterations are the most efficient and faster than the existing ones of similar nature.

Author Contributions: Development and original draft preparation, T. Zhanlav; writing-original draft preparation, Kh. Otgondorj; manuscript review and editing, V. Ulziibayar; writing-review and numerical experiments, Kh. Enkhbayar. All authors have reviewed and approved the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data sharing is not applicable.

Acknowledgments: The authors wish to thank the editor and the anonymous referees for their valuable suggestions and comments on the first version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

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УДК 519.872, 519.217
PACS 07.05.Tp, 02.60.Pn, 02.70.Bf
DOI: 10.22363/2658-4670-2026-34-1-40-54 EDN: VEJQIO

Итерации без производных в R^n с поточечными операциями для решения систем нелинейных уравнений

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Аннотация. В данной работе мы разрабатываем новое семейство итерационных методов высокого порядка без использования производных для решения систем нелинейных уравнений. В частности, мы предлагаем четыре двухшаговые схемы без использования производных с порядками сходимости четыре и пять, а также двенадцать трёхшаговых схем без использования производных, достигающих порядков сходимости шесть, семь и восемь. Главная особенность этих итераций заключается в том, что они включают векторный или даже скалярный параметр итерации вместо матричного параметра, присущего другим существующим итерационным методам. Это структурное упрощение значительно снижает вычислительные затраты, требования к хранению данных и матричные операции, тем самым повышая общую вычислительную эффективность. Представлен анализ сходимости, устанавливающий теоретический порядок сходимости предлагаемых методов. Выведены показатели эффективности предложенных схем и проведено их сравнение с показателями нескольких известных итерационных методов без использования производных. Численные эксперименты на стандартных академических задачах подтверждают теоретические результаты и демонстрируют, что предложенные методы являются конкурентоспособными и во многих случаях превосходят другие методы с точки зрения эффективности и устойчивости.

Ключевые слова: нелинейные системы, итерации без производных, индекс эффективности, порядок сходимости