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The waiting time extremal index in GI/G/1 system

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Abstract. In this paper the conditions to compare the extremal index of the stationary waiting time in the $M/G/1$ and $GI/M/1$ systems are obtained. These conditions include exponential asymptotic behaviour of waiting time tail and the order in failure rates for the interarrival intervals and for the service times in the systems to be compared. For $M/G/1$ system the obtained result is extended to the mixed service times with ordered components. If, in a $GI/G/1$ system, the service time is determined by a finite mixture whose dominant component of the equilibrium distribution belongs to the class of subexponential distributions then the tail of the limiting distribution of the stationary waiting time is equivalent to the tail of this distribution up to a constant obtained explicitly. Furthermore, the limiting distribution of the maximum of the stationary waiting time belongs to the maximum domain of attraction of the distribution of extreme values of the same type as the maximum of the random variables defined by the dominant component.

Key words and phrases: extremal index, queueing system, order in failure rate

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1. Introduction

Understanding whether extreme events happen independently or in groups is crucial for forecasting them and minimizing their impact. Extreme value theory can accommodate clustering via so-called *extremal index* θ which, measures the size of the cluster and thus has an appealing physical meaning [1–4].

When $\theta = 1$, extremes behave like a Poisson process, occurring in isolation. However, when $\theta < 1$, extremes occur in groups (form clusters), following a compound Poisson process. In this case, $1/\theta$ estimates the mean cluster size or, equivalently, the mean time spent above the threshold. The limiting distribution of the maximum value in a stationary sequence is directly shaped by θ, revealing the local dependence within the data.The extremal index therefore gives a measure of the fraction of extremes that are approximately independent and identically distributed (i.i.d.) [5]. The extreme case of $\theta = 0$ indicates total dependence, where exceedances form very wide clusters. In practice, this means a sufficiently high threshold may never be crossed. Conversely, independent sequences always have $\theta = 1$, with high thresholds exceeded only by isolated events.

It is important to mention that there are other definitions of clustering and extremal index in the literature [6, 7].

From a practical point of view, the extremal index is useful for estimating the size of clusters or the average length of intervals between the exceedances. In the telecommunications the interest is the estimation of the risk to lose customers with maximum waiting times (deadlines) exceeding the threshold. The study of extremal metrics in queueing theory relies on applying extreme value theory to regenerative processes. A central objective is determining the limiting distribution for the maxima of waiting times, virtual waiting times, or queue lengths, see for example, [8–11]. The extreme value theory for independent and *one-dependent* regeneration processes is developed in [11]. The algorithm of computing the extremal index of the stationary waiting time of a stable $G/G/1$ system with distribution belonging to the domain of attraction of Gumbel distribution is given in [9]. The study of the distribution of the cluster and inter-cluster sizes is also an actual problem (see, for example, results for the Lindley process in $GI/G/1$ in [12]).

For a stable $GI/G/1$ system, if a non-zero solution $\gamma > 0$ exists for the equation $e^{\gamma(S-\tau)} = 1$, the maximum waiting time converges to a Gumbel distribution. The extremal index θ in this case is often amenable to explicit or numerical computation.

For the case of subexponential distributions of service times distributions, the parameter $\gamma = 0$, and the asymptotics of extreme values are studied using alternative methods based on the tail behavior of the waiting times themselves.

The main contribution of this paper is to establish conditions under which the extremal indexes of two queuing systems can be compared. Another objective is to determine which monotonicity properties in terms of the extremal index can be established for the systems with mixed service times.

The paper is organized as follows. In Section 2, we present known results on the extremal index for $GI/M/1$ and $M/G/1$ systems. In Section 3 Theorems 1 and 2 were proved. They establish the conditions for comparing the extremal indexes of stationary waiting times in $M/G/1$ and $GI/M/1$, respectively. For $M/G/1$ system the obtained result is extended to the case of a system with mixed service times with ordered components (Section 3.1). In Section 3.2 we extend results obtained for $GI/M/1$ system to multiserver $GI/M/c$. In Section 4 we investigate the class of limiting distributions of the stationary waiting time in $GI/G/1$ system with service time determined by a finite mixture whose dominant component of the equilibrium distribution belongs to the subexponential distributions.

2. The extremal index in GI/G/1 system

Let $GI/G/1$ system have i.i.d. service times, $\{S_i, i \geqslant 1\}$ and i.i.d. interarrivals, $\{\tau_i, i \geqslant 1\}$. Consider a reflected random walk (Lindley process), given by the recursion

$$W_{i+1} = (W_i + X_i)^+, \quad i \geqslant 1, \tag{1}$$

where $(x)^+ = \max(0, x)$ and $X_i = S_i - \tau_i$ — i.i.d. non-lattice r.v.s with common distribution function (d.f.) F_X, $\mathrm{E}X < 0$. Also assume that there is a γ such that

$$\mathrm{E}e^{\gamma X} = 1. \tag{2}$$

Let $Y_n = X_1 + \dots + X_{n-1}$, $n \geqslant 1$ ($Y_0 = 0$). The actual waiting time W_n (or just waiting time) of customer n is the time from arrival t_n to the system until service starts. This process $\{W_n, n \geqslant 1\}$ is a Lindley process (1) generated by $\{Y_n, n \geqslant 0\}$ [8]. In particular, $W_n \stackrel{d}{=} \max\limits_{0 \leqslant k \leqslant n} Y_k$ and, if $\rho = \mathrm{E}S/\mathrm{E}\tau < 1$ (corresponds to $\mathrm{E}X < 0$), then a limiting steady-state distribution exists. By W we denote a random variable having the steady-state distribution of actual waiting time process $\{W_n\}$,

$$W_n \Rightarrow W, \quad n \to \infty.$$

It is obvious that the behaviour of the tail of $\mathrm{P}(\max Y_n > x)$ is determined by the components of X_i. Next we consider two cases: the real positive root of the equation (2) $\gamma > 0$. This case refers to light-tailed distributions of X_i and exponential asymptotic behaviour of waiting time tail. In contrast, for subexponential distributions of X_i we get $\mathrm{E}e^{\gamma X} = \infty$ for any $\gamma > 0$ [8].

Let $\overline{\mathrm{F}_S}(x) = 1 - \mathrm{F}_S(x)$ be the tail of d.f. $\mathrm{F}_S(x)$ of r.v. S. A d.f. F_S is called *subexponential* if

$$\lim_{x\to\infty} \frac{\overline{\mathrm{F}_S^{*n}}(x)}{n\overline{\mathrm{F}_S}(x)} = 1 \quad \text{for all } n \geqslant 2,$$

where $\overline{\mathrm{F}_S^{*n}}(x)$ is the tail of the n-fold convolution of the distributions $\mathrm{F}_S(x)$ with itself, i.e., $\overline{\mathrm{F}_S^{*n}}(x) = \mathrm{P}(S^1 + \dots + S^n > x)$, where S^i is the stochastic copy of S, $i = 1, \dots, n$.

We denote the class of subexponential distributions by $\mathcal{S}$. We also denote by S_e the stationary residual service time given by the probability density function $\overline{\mathrm{F}_S}(x)/\mathrm{E}S$, and let $\mathrm{F}_{S_e}(x)$ be the d.f. of S_e.

It is known [8, 10] that if $\rho < 1$ the system is stationary, and equation (2) has a real positive solution $\gamma > 0$ [13], then:

1) if $\mathrm{E}[Xe^{\gamma X}] < \infty$, then the tail d.f. of $\max\limits_{n \geq 0} Y_n$ is asymptotically (up to certain constant $K > 0$) equivalent to an exponential function, namely:

$$\lim_{x\to\infty} \frac{\mathrm{P}(\max\limits_{n\geqslant 0} Y_n > x)}{e^{-\gamma x}} = K.$$

2) if $\mathrm{E}[Xe^{\gamma X}] = \infty$, then $\lim\limits_{x\to\infty} P(\max\limits_{n\geqslant 0} Y_n > x) = \mathrm{o}(e^{-\gamma x})$, as $x \to \infty$, where $b = \mathrm{o}(a)$ means $\lim b/a = 0$.

If $S_e \in \mathcal{S}$, then the the stationary waiting time W is also subexponential, $W \in \mathcal{S}$, and [13]

$$\lim_{x\to\infty} \frac{\mathrm{P}(\max\limits_{n\geqslant 0} Y_n > x)}{\mathrm{P}(S_e > x)} = \frac{\rho}{1-\rho}. \tag{3}$$

Denote by $M_n = \max(W_1, \dots, W_n)$ the largest waiting time among customers $1, \dots, n$. If $\rho \leqslant 1$ and there exists a real positive solution to equation (2), then [14–16]

$$\lim_{n\to\infty} \mathrm{P}(\gamma M_n - \log(b\theta n) \leqslant x) = \Lambda(x), \tag{4}$$

where $\Lambda(x) = \exp(-e^{-x})$ is the Gumbel distribution and b is a constant. Moreover, $M_n/\log n$ converges to $1/\gamma$ whenever possible [15] for all $\epsilon > 0$:

$$\mathrm{P}\left(\left|\frac{M_n}{\log n} - \frac{1}{\gamma}\right| > \frac{\epsilon}{\gamma}\right) \to 0 \quad \text{as } n \to \infty.$$

For the $GI/M/1$ system with $\rho < 1$, equation (2) gives a unique positive solution γ [8, Theorem 5.8]. Moreover, the distribution of the actual waiting time W is a mixture of an atom at zero and an exponential distribution with parameter γ and mixture proportions γ/μ and $1 - \gamma/\mu$, respectively, [8, Theorem 5.1]:

$$\mathrm{F}_W(x) = 1 - (1 - \gamma/\mu)e^{-\gamma x}, \quad x > 0.$$

Theorem 7.5 in [15] implies that the extremal index of the process $\{W_n, n \geqslant 1\}$ for a $GI/M/1$ system is calculated by the formula:

$$\theta = \gamma\left(\frac{d}{d\gamma}\psi_\tau(\gamma) + \frac{1}{\mu}\right), \tag{5}$$

where $\psi_\tau(\gamma) = \mathrm{E}e^{-\gamma\tau}$ is the Laplace-Stieltjes transform of interarrival times, τ, and, in addition, $b = 1 - \gamma/\mu$ in (4).

For a stationary $M/G/1$ system, equation (2) has the following form [8]:

$$\mathrm{E}e^{\gamma S} = 1 + \frac{\gamma}{\lambda}. \tag{6}$$

Furthermore, if equation (6) has a real positive solution γ, then the formula for the extremal index θ_W becomes [9]:

$$\theta = \mathrm{P}(W = 0)(1 - \psi_\tau(\gamma)) = \frac{\gamma(1-\rho)}{\gamma + \lambda}. \tag{7}$$

3. Comparison of extremal indexes in $GI/G/1$ systems

Consider two queuing systems $\Sigma^{(1)}$ and $\Sigma^{(2)}$ of type $GI/G/1$ (we assign index i to quantities related to the i-th system). Let $S_n^{(i)}$ be the service time of the n-th customer and $\tau_n^{(i)}$ be the interval between the arrivals of the n-th and $n+1$-th requests in the i-th system, $\mathrm{E}\tau^{(i)} = 1/\lambda$, $i = 1, 2$. Let $W_n^{(i)}$ be the actual waiting time of n-th customer, $i = 1, 2$. Let us denote (if they exist) the distribution limits

$$W_n^{(i)} \Rightarrow W^{(i)}, \quad n \to \infty, \quad i = 1, 2.$$

These limits exist, in particular, if the interarrival times $\tau^{(i)}$, $i = 1, 2$ are non-lattice and $\rho_i = \lambda_i \mathrm{E}S^{(i)} < 1$, $i = 1, 2$ [8].

Let us compare the extremal indexes $\theta^{(1)}$ and $\theta^{(2)}$ of stationary waiting time processes $\{W_n^{(1)}\}$ and $\{W_n^{(2)}\}$ in the systems $\Sigma^{(1)}$ and $\Sigma^{(2)}$, respectively. Further, to compare the r.v.s, we will need the stochastic order and the order in failure rate. We say that the r.v. Z_1 is less that r.v. Z_2 in stochastic order, $Z_1 \underset{st}{\leqslant} Z_2$ if

$$\overline{\mathrm{F}_{Z_1}}(x) \leqslant \overline{\mathrm{F}_{Z_2}}(x), \quad x \in \mathrm{R}.$$

Let $r_Z(x) := f_Z(x)/\overline{\mathrm{F}_Z}(x)$ be failure rate function of r.v. Z, where $f_Z(x)$ is density function. We say that r.v. Z_1 is less than r.v. Z_2 in failure rate order, $Z_1 \underset{r}{\leqslant} Z_2$, if

$$r_{Z_1}(x) \geqslant r_{Z_2}(x), \quad x \in \mathrm{R}.$$

The following theorem allows to compare the extremal indexes of stationary waiting times in two different $M/G/1$ systems for which the real positive roots of equation (2) exist.

Theorem 1. *Suppose that in two $M/G/1$ systems $\Sigma^{(1)}$ and $\Sigma^{(2)}$, $\rho_i < 1$, $\mathrm{E}[S^{(i)}e^{\gamma S^{(i)}}] < \infty$, $i = 1, 2$ for and any $\gamma \geqslant 0$ and relations*

$$W_1^{(1)} = W_1^{(2)} = 0, \quad \tau^{(1)} \underset{r}{\geqslant} \tau^{(2)}, \quad S^{(1)} \underset{r}{\leqslant} S^{(2)}, \tag{8}$$

are satisfied. Additionally, suppose that there exist real positive roots γ_1 and γ_2 of equation (2) *for both systems. Then the extremal indices of the stationary waiting times are ordered,*

$$\theta^{(1)} \geqslant \theta^{(2)}. \tag{9}$$

Proof. First of all, note that the ordering in failure rate implies the ordering of the exponential moments [17]. Consequently, from the relations (8) it follows that

$$\mathrm{E}e^{\gamma S^{(1)}} \leqslant \mathrm{E}e^{\gamma S^{(2)}}, \quad \mathrm{E}e^{-\gamma\tau^{(1)}} \leqslant \mathrm{E}e^{-\gamma\tau^{(2)}}, \quad \text{for any } \gamma > 0.$$

Let γ_i denote the positive real root of equation (2) for the system $\Sigma^{(i)}$, $i = 1, 2$. If there are several such roots, then, according to Theorem 7.2 in [15], the smallest of them is chosen. Then the equalities

$$\mathrm{E}e^{\gamma_1(S^{(1)}-\tau^{(1)})} = \mathrm{E}e^{\gamma_2(S^{(2)}-\tau^{(2)})} = 1$$

are satisfied if

$$\gamma_1 \geqslant \gamma_2.$$

Since the intervals between arrivals are ordered in failure rate, $\tau^{(1)} \underset{r}{\geqslant} \tau^{(2)}$, then $\lambda_1 < \lambda_2$. Consequently, $\lambda_1/\gamma_1 \leqslant \lambda_2/\gamma_2$. Moreover, by $S^{(1)} \underset{r}{\leqslant} S^{(2)}$, we also have $\mathrm{E}S^{(1)} \leqslant \mathrm{E}S^{(2)}$ and $1-\rho_1 \geqslant 1-\rho_2$. Substituting these inequalities into the expression for the extremal index (7), we obtain

$$\theta^{(1)} = \frac{1-\rho_1}{1+\lambda_1/\gamma_1} \geqslant \frac{1-\rho_2}{1+\lambda_2/\gamma_2} = \theta^{(2)}.$$

To illustrate the statement of Theorem 1 we consider the simple example. It's easy to show that, for the $M/M/1$ system, the extremal index of the stationary waiting time has the form

$$\theta = (1-\rho)^2.$$

Consider two $M/M/1$ systems in which the service times are exponential with parameters μ_1 and μ_2, respectively, and $\mu_1 \geqslant \mu_2 > 0$. Assume that $\lambda_1 \leqslant \lambda_2$ and $\rho_i = \lambda_i/\mu_i < 1$, $i = 1, 2$. In this case, the conditions (8) are satisfied and

$$\gamma_1 = \mu_1 - \lambda_1 \geqslant \gamma_2 = \mu_2 - \lambda_2, \theta^{(1)} = (1-\rho_1)^2 \geqslant (1-\rho_2)^2 = \theta^{(2)},$$

i.e., the inequality (9) is satisfied.

Now consider $M/We/1$ system with Weibull service time d.f.

$$\mathrm{F}_S(x) = 1 - e^{-(x/\alpha)^\beta}, \quad \alpha,\ \beta > 0, \quad x \geqslant 0.$$

The exponential moments $\mathrm{E}e^{\gamma S}$ for the Weibull distribution exist only for $\beta \geqslant 1$, so in this case the equation (6) takes form

$$\sum_{k=0}^{\infty} \frac{(\alpha\gamma)^k}{k!}\Gamma\left(\frac{k}{\beta}+1\right) = 1 + \frac{\gamma}{\lambda}, \quad \text{for } \beta \geqslant 1.$$

Now we compare the extremal indexes of waiting times in two $M/We/1$ systems. For example, let $\lambda = 2$, $\alpha_1 = 0.25$, $\beta_1 = 1.5$ in the first system and $\lambda_2 = 2$, $\alpha_2 = 0.4$, $\beta_2 = 1.2$. With these parameters we have $S^{(1)} \underset{r}{<} S^{(2)}$. By numerical calculation we obtain a single roots $\gamma_1 = 3.7$ and $\gamma_2 = 0.79$, respectively, therefore, $\theta^{(1)} = 0.35 > \theta^{(2)} = 0.07$.

Now we prove a statement similar to the Theorem 1 for $GI/M/1$ systems.

Theorem 2. *Let the stationarity conditions $\rho_i < 1$, $\mathrm{E}[(S^{(i)}e^{\gamma S^{(i)}}] < \infty$ for any $\gamma \geqslant 0$, $i = 1, 2$, and the relations* (8) *be satisfied for two $GI/M/1$-type systems $\Sigma^{(1)}$ and $\Sigma^{(2)}$. Let there exist real positive roots γ_1 and γ_2 of the equation* (2) *for these systems and the following inequality holds:*

$$\mathrm{E}[\tau^{(1)}e^{-\gamma_1\tau^{(1)}}] \leqslant \mathrm{E}[\tau^{(2)}e^{-\gamma_2\tau^{(2)}}]. \tag{10}$$

Then the extremal indexes of the stationary waiting times are ordered as

$$\theta^{(1)} \geqslant \theta^{(2)}.$$

Proof. In the proof of Theorem 1 it is shown that the roots γ_i of equation (2) for systems $\Sigma^{(i)}$ are related by the inequality

$$\gamma_1 \geqslant \gamma_2,$$

and by (8) the exponential moments are also ordered in the same way. Therefore,

$$\psi_{\tau^{(1)}}(\gamma_1) \leq \psi_{\tau^{(2)}}(\gamma_2).$$

Note, that $\mathrm{E}\tau e^{-\gamma\tau} < \infty$ for any r.v. $\tau \geq 0$ and any $\gamma \geq 0$ and that

$$\frac{d}{d\gamma_i}\psi_{\tau^{(i)}}(\gamma_i) = -\mathrm{E}(\tau^{(i)}e^{-\gamma_i\tau^{(i)}}), \quad i = 1, 2.$$

Thus, it follows from condition (10), that

$$\frac{d}{d\gamma_1}\psi_{\tau^{(1)}}(\gamma_1) \geqslant \frac{d}{d\gamma_2}\psi_{\tau^{(2)}}(\gamma_2).$$

The equation (2) for the systems under consideration is equivalent to

$$\psi_{\tau^{(i)}}(\gamma_i) + \gamma_i/\mu_i = 1, \quad i = 1, 2.$$

Therefore, expression (5) can be rewritten as

$$\theta^{(i)} = \gamma_i(\frac{d}{d\gamma_i}\psi_{\tau^{(i)}}(\gamma_i) + 1/\mu_i) = (1 - \psi_{\tau^{(i)}}(\gamma_i))(\mu_i\frac{d}{d\gamma_i}\psi_{\tau^{(i)}}(\gamma_i) + 1), \quad i = 1, 2.$$

From the ordering in failure rate of service times, it follows that $\mu_1 \geqslant \mu_2$. Substituting the obtained inequalities into the expression for the extremal index (5), we obtain the required inequality

$$\theta^{(1)} = (1 - \psi_{\tau^{(1)}}(\gamma_1))(\mu_1\frac{d}{d\gamma_1}\psi_{\tau^{(1)}}(\gamma_1) + 1) \geqslant (1 - \psi_{\tau^{(2)}}(\gamma_2))(\mu_2\frac{d}{d\gamma_2}\psi_{\tau^{(2)}}(\gamma_2) + 1) = \theta^{(2)}.$$

It is worth noting that the set of queueing systems satisfying the inequality (10) is not empty. In particular it holds for the popular $M/M/1$ system because

$$\frac{d}{d\gamma_i}\psi_{\tau^{(i)}}(\gamma_i) = -\frac{\lambda_i}{(\lambda_i + \gamma_i)^2} = -\frac{\lambda_i}{\mu_i^2},$$

and if $\lambda_1 \leqslant \lambda_2, \mu_1 \geqslant \mu_2$, then condition (10) holds. As another example, consider two systems $\Sigma^{(1)}$ and $\Sigma^{(2)}$, with two-component hyperexponential interarrival times, where d.f. tail has form

$$\overline{\mathrm{F}_{\tau^{(i)}}}(x) = pe^{-\lambda_1^{(i)}x} + (1-p)e^{-\lambda_2^{(i)}x}, \quad \lambda_j^{(i)} > 0, \quad 0 < p < 1, \quad x \geqslant 0, \quad i, j = 1, 2.$$

Let $p = 0.5, \lambda_1^{(1)} = 1$, $\lambda_2^{(1)} = 3$, $\lambda_1^{(2)} = 2$, $\lambda_2^{(2)} = 3$. Suppose that the service time is exponential with parameter $\mu_1 = 4$ in $\Sigma^{(1)}$ and with parameter $\mu_2 = 3$ in $\Sigma^{(2)}$. Then numerical analysis shows that only positive (numerical) solution of (2) is $\gamma_1 = 2.24$, for which $\theta_{W^{(1)}} = 0.33$. Similarly, the unique positive solution $\gamma_2 = 0.58$ can be obtained by solving numerically the equation (2), for which $\theta_{W^{(2)}} = 0.0385$. With these parameters, the conditions (8) and (10) are satisfied, since

$$r_{\tau^{(1)}}(x) = 1 < r_{\tau^{(2)}}(x) = 2; \qquad r_{S^{(1)}}(x) = 4 > r_{S^{(2)}}(x) = 3,$$

and

$$\gamma_1 > \gamma_2, \quad \theta^{(1)} > \theta^{(2)}.$$

3.1. $M/G/1$ system with mixed service times

Consider a single-server $M/G/1$-type system Σ with service time S given by an m-component mixture d.f.

$$\mathrm{F}_S(x) = \sum_{i=1}^{m} p_i \mathrm{F}_{S^{(i)}}(x), \quad \sum_{i=1}^{m} p_i = 1, \quad p_i \geqslant 0, \quad i = 1, \dots, m. \tag{11}$$

Assume that the r.v.'s $S^{(1)}, \dots, S^{(m)}$ are independent and $S^{(i)}$ has d.f. $\mathrm{F}_{S^{(i)}}(x)$, $i = 1, \dots, m$. Denote by

$$\rho = \lambda \mathrm{E}S = \sum_{i=1}^{m} \frac{\lambda p_i}{\mu_i} = \sum_{i=1}^{m} p_i \rho_i$$

the traffic intensity of the system Σ where $\rho_i = \lambda/\mu_i$, $\mu_i = 1/\mathrm{E}S^{(i)}$, $i = 1, \dots, m$. Assume that the components $S^{(i)}$ of the service time S are ordered in failure rate

$$S^{(1)} \underset{r}{\leqslant} \cdots \underset{r}{\leqslant} S^{(m)}.$$

Consider two queuing systems $\Sigma^{(1)}$ and $\Sigma^{(m)}$ with inputs $\tau^{(1)}$, $\tau^{(m)}$, respectively. Let τ be the input process in the original system Σ, and $\mathrm{E}\tau^{(i)} = 1/\lambda_i$, $\mathrm{E}\tau = 1/\lambda$, $i = 1, m$. (As usual, the index i relates to the i-th system.) The service time $S^{(i)}$ is given by the d.f. $\mathrm{F}_{S^{(i)}}(x)$, $i = 1, m$.

Theorem 3. *Assume the stationarity conditions $\rho_i < 1$, $i = 1, m$; $\rho < 1$ hold and the following relations are satisfied:*

$$W_1^{(1)} = W_1^{(m)} = W_1 = 0.$$

$$\tau^{(1)} \underset{r}{\geqslant} \tau \underset{r}{\geqslant} \tau^{(m)}.$$

Suppose, that the components of the service time mixture are ordered

$$S^{(1)} \underset{r}{\leqslant} \cdots \underset{r}{\leqslant} S^{(m)}.$$

Assume that real positive roots of equation (2) *exist for all three systems. Then*

$$\theta^{(1)} \geqslant \theta \geqslant \theta^{(m)}.$$

The proof follows from Theorem 1 and the monotonicity property of waiting times (see Theorems 5 and 6 in [18]).

To illustrate the statement of Theorem 3, consider an $M/H_m/1$ system as the original system where service times have m-component mixture d.f.

$$\overline{\mathrm{F}_S}(x) = \sum_{i=1}^{m} p_i e^{-\mu_i x}, \quad \mu_i > 0, \quad \sum_{i=1}^{m} p_i = 1, p_i \geqslant 0, \quad x \geqslant 0.$$

We consider two systems $\Sigma^{(1)}$ and $\Sigma^{(m)}$ (of type $M/M/1$), in which the service times $S^{(i)}$ have exponential distribution, $\overline{\mathrm{F}_{S^{(i)}}}(x) = e^{-\mu_i x}$, $i = 1, m$. The interarrival times in all three systems have exponential distribution with parameter λ. Assume that the stationarity conditions are satisfied in all systems:

$$\rho_i = \lambda/\mu_i < 1, \quad i = 1, m, \quad \rho = \lambda \sum_{i=1}^{m} p_i/\mu_i < 1.$$

The service time failure rate function r_S in the original system Σ is equal to

$$r_S(x) := \frac{\sum\limits_{i=1}^{m} p_i \mu_i e^{-\mu_i x}}{\sum\limits_{i=1}^{m} p_i e^{-\mu_i x}}, \quad x \geqslant 0.$$

The service time failure rate functions $S^{(1)}$ and $S^{(m)}$ in the systems $\Sigma^{(1)}$ and $\Sigma^{(m)}$ are, respectively, equal to $r_{S^{(1)}}(x) = \mu_1$, $r_{S^{(m)}}(x) = \mu_m$. It is easy to verify that for

$$\mu_1 \geqslant \cdots \geqslant \mu_m \tag{12}$$

the failure rate functions are ordered as follows:

$$r_{S^{(1)}}(x) \geqslant r_S(x) \geqslant r_{S^{(m)}}(x), \quad x \geqslant 0,$$

and, therefore, the service times in these systems are ordered in the failure rate as

$$S^{(1)} \underset{r}{\leqslant} S \underset{r}{\leqslant} S^{(m)}.$$

The extremal index in the original system Σ can be calculated by the formula (7). Moreover, from the condition for the parameters (12) and the equation (2) it follows that

$$\frac{\lambda+\gamma}{\lambda} = \sum_{i=1}^{m} \frac{p_i \mu_i}{\mu_i - \gamma} \geqslant \sum_{i=1}^{m} \frac{p_i}{1-\gamma/\mu_1} = \frac{\mu_1}{\mu_1 - \gamma},$$

and, therefore,

$$\gamma(\mu_1 - \lambda - \gamma) \geqslant 0$$

and $\gamma \leqslant \mu_1 - \lambda = \gamma_1$. Further, since $\rho_1 \leqslant \rho$, then

$$\theta = \frac{1-\rho}{1+\lambda/\gamma} \leqslant \frac{1-\rho_1}{1+\lambda/\gamma_1} = (1-\rho_1)^2 = \theta^{(1)}.$$

Similarly, it can be shown that

$$\frac{\lambda+\gamma}{\lambda} = \sum_{i=1}^{m} \frac{p_i \mu_i}{\mu_i - \gamma} \leqslant \frac{\mu_m}{\mu_m - \gamma},$$

and $\gamma \leqslant \mu_m - \lambda = \gamma_m$, and therefore,

$$\theta = \frac{1-\rho}{1+\lambda/\gamma} \geqslant \frac{1-\rho_m}{1+\lambda/\gamma_m} = (1-\rho_m)^2 = \theta^{(m)}.$$

3.2. Extension of Theorem 2 to multiserver systems

The extremal behaviour of multiserver system $GI/M/c$ with c working in parallel servers and service intensity μ is the same as the in $GI/M/1$ system with service intensity $c\mu$ with identical interarrival distribution $\mathrm{F}_\tau(x)$ [9]. Moreover, if $\rho = (\mu \mathrm{E}\tau)^{-1} < c$, then waiting time process has (total variation) limit which is a mixture of an atom at zero and exponential distribution with intensity γ, where γ is unique solution of the equation [8][Theorem 3.2]

$$\int_0^\infty e^{-\gamma x} \mathrm{F}_\tau(dx) = 1 - \frac{\gamma}{c\mu}.$$

From (5) one can also easily obtain the relation for extremal index of stationary waiting time in $GI/M/c$ system by substituting $c\mu$ instead μ, namely,

$$\theta = \gamma\left(\frac{d}{d\gamma}\psi_\tau(\gamma) + \frac{1}{c\mu}\right).$$

The last formula allows us to compare the extremal indexes of waiting times in two $GI/M/c_i$ systems by adding the condition $c_1 \geqslant c_2$ in Theorem 2. We formulate the obtained result as the following statement.

Theorem 4. *Let the stationarity conditions, $\rho_i = \lambda_1 \mathrm{E}S^{(i)} < c_i$, $i = 1, 2$, and the relations* (8)*, be satisfied for two systems $\Sigma^{(1)}$ and $\Sigma^{(2)}$ of type $GI/M/c_i$. Let there exist real positive roots of the equation* (2) *for these systems and*

$$c_1 \geqslant c_2.$$

Then the extremal indexes of the stationary waiting times are ordered as

$$\theta^{(1)} \geqslant \theta^{(2)}.$$

In particular, it is easy to check that for $M/M/c$ system

$$\gamma = c\mu - \lambda, \theta = \left(1 - \frac{\lambda}{c\mu}\right)^2.$$

Therefore, if $\lambda_1 \leqslant \lambda_2$, $\mu_1 \geqslant \mu_2$, $c_1 \geqslant c_2$, then $\theta^{(1)} \geqslant \theta^{(2)}$.

4. GI/G/1 system with subexponential service times

In this section we consider the systems with the subexponential service times. In contrast to the light-tailed case, the stationary waiting time W is also subexponential and the relation (3) holds. Moreover, the maximum stationary waiting time $M_n = \max(W_0, \dots, W_n)$ has the same asymptotics as $\max(X_0, \dots, X_n)$, as $n \to \infty$, with $X_i = S_i - \tau_i$ [19]. Moreover, if $S \in \mathcal{S}$, then the extremal index of the stationary waiting time for systems with subexponential service time is zero [19], i.e.,

$$\theta = 0.$$

Now consider $GI/G/1$ system with m-component mixture service times with d.f. given by (11) with a dominant component.

We say that a component $\mathrm{F}^{(j)}$, $j \in \{1, \dots, m\}$ is asymptotically r-dominant for an m-component mixture of distributions

$$\mathrm{F}(x) = p_1\mathrm{F}^{(1)}(x) + \dots + p_m\mathrm{F}^{(m)}(x), \qquad \sum_{i=1}^{m} p_i = 1,$$

if

$$\lim_{x \to x_R} \frac{\overline{\mathrm{F}^{(i)}}(x)}{\overline{\mathrm{F}^{(j)}}(x)} = r_i, \quad i \in \{1, \dots, m\}, \quad i \neq j,$$

where $r = (r_1, \dots, r_m)$, $r_j = 1$, and $0 \leqslant r_i < 1$, for $i \neq j$, x_R is the right endpoint of $\mathrm{F}(x)$.

The following theorem states that, if in a $GI/G/1$ system, the service time is determined by a finite mixture (11) whose r-dominant component of the equilibrium distribution belongs to the class of subexponential distributions, $\mathrm{F}_{S_e^{(j)}} \in \mathcal{S}$, then the tail of the limit distribution of the stationary waiting time is equivalent to the tail of this distribution up to a constant,

$$\lim_{x\to\infty} \frac{\mathrm{P}(W > x)}{\overline{\mathrm{F}_{S_e^{(j)}}}(x)} =: \delta.$$

Furthermore, the limit distribution of the maximum of the stationary waiting time belongs to the maximum domain of attraction (MDA) of the distribution of extreme values of the same type as the maximum of the r.v. defined by the d.f. $\mathrm{F}_{S_e^{(j)}}$, and the extremal index of the stationary waiting time is obviously 0.

Theorem 5. *Let the original system Σ be stationary, $\rho < 1$. Let the service time be defined by an m-component mixture of distributions* (11) *and let there exist a set of numbers*

$$r = (r_1, \dots, r_m), \quad 0 \leqslant r_i < 1, \quad i \neq j, \quad r_j = 1,$$

such that the equilibrium distribution of the j-th component is r-dominant in the mixture and belongs to the class of subexponential distributions, $\mathrm{F}_{S_e^{(j)}} \in \mathcal{S}$. Then

1) The equilibrium distribution of service time belongs to the class of subexponential distributions, $\mathrm{F}_{S_e} \in \mathcal{S}$.

2) The tails of the equilibrium distributions F_{S_e} and $\mathrm{F}_{S_e^{(j)}}$ are equivalent up to a constant,

$$\lim_{x\to\infty} \frac{\overline{\mathrm{F}_{S_e}}(x)}{\overline{\mathrm{F}_e^{(j)}}(x)} = \sum_{i=1}^{m} q_i r_i$$

where

$$r_i := \lim_{x\to\infty} \frac{\overline{\mathrm{F}_{S_e^{(i)}}}(x)}{\overline{\mathrm{F}_{S_e^{(j)}}}(x)}, \quad 0 \leqslant r_i < 1, \quad i = 1, \dots, m, \quad i \neq j, \quad r_j = 1;$$

$q_i = \dfrac{p_i \mathrm{E}S^{(i)}}{\mathrm{E}S}.$

3) The tails of the distribution of the stationary waiting time, $\mathrm{P}(W > x)$, and the equilibrium distribution $\mathrm{F}_{S_e^{(j)}}$ are equivalent up to a constant δ,

$$\lim_{x\to\infty} \frac{\mathrm{P}(W > x)}{\overline{\mathrm{F}_{S_e^{(j)}}}(x)} = \frac{\lambda \sum\limits_{i=1}^{m} p_i r_i \mathrm{E}S^{(i)}}{1 - \lambda \sum\limits_{i=1}^{m} p_i \mathrm{E}S^{(i)}} := \delta.$$

4) If $\mathrm{F}_{S_e^{(j)}} \in MDA(G)$, then d.f. stationary waiting time in the original system $\mathrm{F}_W \in MDA(G^{\delta})$.

5) The extremal index of the stationary waiting time in the original system is zero, $\theta_W = 0$.

Proof. 1) Find the equilibrium distribution of service time in the original system

$$\overline{\mathrm{F}_{S_e}}(x) = \frac{1}{\mathrm{E}S} \int\limits_x^{\infty} (p_1 \overline{\mathrm{F}_{S^{(1)}}}(y) + \dots p_m \overline{\mathrm{F}_{S^{(m)}}}(y))dy = \sum_{i=1}^{m} \frac{p_i}{\mathrm{E}S} \int\limits_x^{\infty} \overline{\mathrm{F}_{S^{(i)}}}(y)dy =$$

$$= \sum_{i=1}^{m} \frac{p_i \mathrm{E}S^{(i)}}{\mathrm{E}S} \overline{\mathrm{F}_{S_e^{(i)}}}(x) = \sum_{i=1}^{m} q_i \overline{\mathrm{F}_{S_e^{(i)}}}(x).$$

Obviously, $\sum_{i=1}^{m} q_i = 1$. Thus, the equilibrium service time distribution in the original system is a mixture of the equilibrium distributions of the components with mixture proportions q_i,

$$S_e = J_1 S_e^{(1)} + \dots + J_m S_e^{(m)}, \quad \sum_{i=1}^{m} J_i = 1$$

where the indicator J_i takes the value 1 with probability q_i, $i = 1, \dots, m$. It is known that if at least one component of the final mixture has a subexponential distribution, then the mixture of distributions belongs to the class of subexponential distributions [20]. Therefore, point 1) of the theorem is proved.

2) Since F_{S_e} is an m-component mixture with proportionality coefficients q_i, where the j-th component is r-dominant, then F_{S_e} and $\mathrm{F}_{S_e^{(j)}}$ have equivalent tails up to the constant $\sum_{i=1}^{m} q_i r_i$ [21].

3) By relation 3 and point 2) of this theorem,

$$\mathrm{P}(W > x) \sim \frac{\rho}{1-\rho} \overline{\mathrm{F}_{S_e}}(x) \sim \frac{\rho}{1-\rho} \sum_{i=1}^{m} q_i r_i \overline{\mathrm{F}_{S_e^{(j)}}}(x) = \delta \overline{\mathrm{F}_{S_e^{(j)}}}(x), \quad \text{as } x \to \infty,$$

where $a \sim b$ means $a/b \to 1$ and

$$\delta = \frac{\rho}{1-\rho} \sum_{i=1}^{m} q_i r_i = \frac{\lambda \sum_{i=1}^{m} p_i r_i \mathrm{E} S^{(i)}}{1 - \lambda \sum_{i=1}^{m} p_i \mathrm{E} S^{(i)}}.$$

The point 4) follows from the point 3) of this theorem and [21, Theorem 8].

5) Since $\mathrm{F}_{S_e} \in \mathcal{S}$ then, for $GI/G/1$ system with subexponential service, the extremal index of the stationary waiting time is zero [19]. □

Corollary 1. Assume that the system Σ is stationary, $\rho < 1$. Let the service time be given by an m-component mixture of distributions (11) with components ordered by the failure rate

$$S^{(1)} \underset{r}{\leqslant} \dots \underset{r}{\leqslant} S^{(m)}. \tag{13}$$

Suppose that $S_e^{(m)} \in \mathcal{S}$. Then all statements of Theorem 5 are true.

Proof. It suffices to show that the distribution $\mathrm{F}_e^{(m)}$ is r-dominant for F_{S_e}. Since the ordering by failure rate of service times (13) implies the stochastic ordering of the r.v. $S_e^{(i)}$,

$$S_e^{(1)} \underset{st}{\leqslant} \dots \underset{st}{\leqslant} S_e^{(m)},$$

that

$$\frac{\overline{\mathrm{F}_{S_e^{(i)}}}(x)}{\overline{\mathrm{F}_{S_e^{(m)}}}(x)} \leqslant 1, \quad \text{for all } x.$$

Obviously, there exist r_i such that

$$\lim_{x \to x_R} \frac{\overline{\mathrm{F}_{S_e^{(i)}}}(x)}{\overline{\mathrm{F}_{S_e^{(m)}}}(x)} = r_i, \quad 0 \leqslant r_i \leqslant 1, \quad i = 1, \dots, m-1, \quad r_m = 1.$$

In this case, the distribution $\mathrm{F}_e^{(m)}$ is asymptotically r-dominant for the mixture distribution F_{S_e}, $r = (r_1, \dots, r_m)$, $r_m = 1$, and the conditions of Theorem 5 are satisfied for $j = m$. □

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As an example, we consider a stationary $GI/G/1$ system (i.e., $\rho = \lambda \mathrm{E}S < 1$) with the service times having an exponential-Pareto distribution [22] with parameter $\alpha > 1$

$$\mathrm{F}_S(x) = 1 - pe^{-\lambda_S x} - (1-p)\left(\frac{x_0}{x_0 + x}\right)^{\alpha}, \quad \lambda_S > 0, \quad \alpha > 1, \quad x_0 > 0, \quad x \geqslant 0.$$

In this case the equilibrium distribution function of S_e has the form

$$\mathrm{F}_{S_e}(x) = \mu \int_0^x \overline{\mathrm{F}_S}(t)dt = 1 - \mu\left(\frac{pe^{-\lambda_S x}}{\lambda_S} + \frac{(1-p)x_0^{\alpha}}{(\alpha-1)(x_0+x)^{\alpha-1}}\right),$$

where $\mu = 1/\mathrm{E}S$. Note that

$$\overline{\mathrm{F}_{S_e}}(x) = q_1\overline{\mathrm{F}_{S_e^{(1)}}}(x) + q_2\overline{\mathrm{F}_{S_e^{(2)}}}(x)$$

and

$$q_1 = \mu p/\lambda_S, \quad q_2 = \mu(1-p)x_0/(\alpha-1).$$

By Theorem 5 the limiting distribution of the maximum stationary waiting time M_n is a Frechet distribution of the form

$$\lim_{n\to\infty} \mathrm{P}(M_n \leqslant u_n(x)) = e^{-\frac{(1-p)\mu\lambda x_0}{(\alpha-1)(\mu-\lambda)}x^{1-\alpha}}$$

with the normalizing sequence (for $x > 0$)

$$u_n(x) = a_n x + b_n = x_0 x n^{1/(\alpha-1)} - x_0, \quad n \geqslant 1.$$

Indeed, it is obvious that the second component of the distribution is r-dominant, with $r_1 = 0, r_2 = 1$,

$$\lim_{x\to\infty} \frac{\overline{\mathrm{F}_{S_e^{(1)}}}(x)}{\overline{\mathrm{F}_{S_e^{(2)}}}(x)} = \lim_{x\to\infty} \frac{e^{-\lambda_S x}}{x_0^{\alpha-1}/(x_0+x)^{\alpha-1}} = 0 = r_1.$$

Now we find maximum domain of attraction of dominant component $S_e^{(2)}$.

$$\mathrm{F}_{S_e^{(2)}}(x) = \frac{\alpha-1}{x_0}\int_0^x \left(\frac{x_0}{x_0+y}\right)^{\alpha} dy = 1 - \left(\frac{x_0}{x_0+x}\right)^{\alpha-1}.$$

Obviously, $S_e^{(2)}$ has a Pareto distribution with parameters $\alpha - 1$, x_0 and therefore belongs to the class of subexponential distributions [23]. Let $v_n(x) = x_0 n^{1/(\alpha-1)}x - x_0$. Then for $n \to \infty$,

$$n\overline{\mathrm{F}_{S_e^{(2)}}}(v_n(x)) = \left(\frac{x_0}{x_0 + x_0 n^{1/(\alpha-1)}x - x_0}\right)^{\alpha-1} \to x^{-\alpha+1}, \quad \text{as } n \to \infty,$$

which implies $\mathrm{F}_{S_e^{(2)}} \in MDA(\Phi_{\alpha-1})$ [1] where $\Phi_\alpha(x) = e^{-x^{-\alpha}}$, $x \geqslant 0$, is Frechet distribution.

Now we can calculate δ

$$\delta = \frac{\lambda(pr_1\mathrm{E}S^{(1)} + (1-p)r_2\mathrm{E}S^{(2)})}{1 - \lambda(p\mathrm{E}S^{(i)} + (1-p)\mathrm{E}S^{(2)})} = \frac{\mu\lambda}{\mu-\lambda}\frac{(1-p)x_0}{\alpha-1}.$$

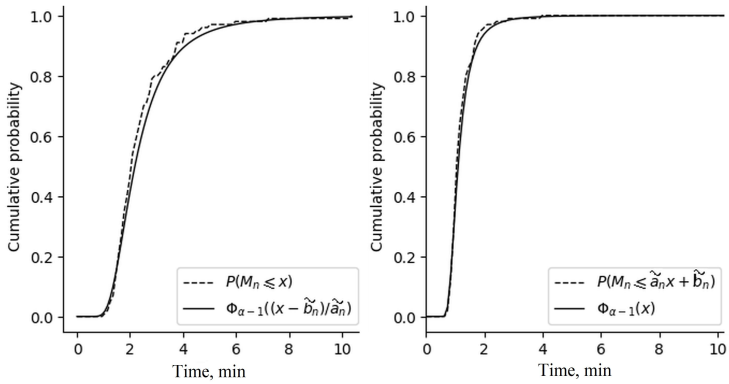

Figure 1. left figure: $\mathrm{P}(M_n \leqslant x)$ and $\Phi_{\alpha-1}(x-\tilde{b}_n)/\tilde{a}_n$, right figure: $\mathrm{P}(M_n \leqslant \tilde{a}_n x+\tilde{b}_n)$ and $\Phi_{\alpha-1}(x)$ for group size $n = 1000$

Since $\mathrm{F}_{S_e^{(2)}} \in MDA(\Phi_{\alpha-1})$, then by 4) of Theorem 5, $\mathrm{F}_W \in MDA(\Phi^{\delta}_{\alpha-1})$.

Thus, the asymptotic behaviour of the maximum stationary waiting time is determined by the second component (the Pareto distribution).

Note also, that since $F_w \in MDA(\Phi^{\delta}_{\alpha-1})$ with $u_n(x) = a_n x + b_n$, then $F_w \in MDA(\Phi_{\alpha-1})$ with $\tilde{u}_n(x) = \tilde{a}_n x + \tilde{b}_n$ where $\tilde{a}_n = a_n \delta^{1/(\alpha-1)}$, $\tilde{b}_n = b_n$.

To illustrate the conclusion of Theorem 5, we carried out a numerical simulation of the system for 100 replications. We compare the estimate of $\mathrm{P}(M_n \leqslant x)$ with $\Phi_{\alpha-1}(x-\tilde{b}_n)/\tilde{a}_n$ and the estimate of $\mathrm{P}(M_n \leqslant \tilde{a}_n x + \tilde{b}_n)$ with $\Phi_{\alpha-1}(x)$. We have run a Kolmogorov–Smirnov (K-S) test for goodness of fit. We find the empirical distribution function for M_n from observed maximum waiting times for $k = 100$ groups of customers (each group of size n). Figure 1 demonstrates results for $x_0 = 1$, $\alpha = 5$, $p = 0,5$, $\lambda_s = 4$, $\lambda = 0.5$ and group size $n = 1000$. K-S test statistic is 0.81 for left figure and 0.74 for right figure. Therefore, our hypothesis that the Frechet distribution with the normalizing constants, a_n, b_n describes nicely the maxima of waiting times (at level 0.05) is confirmed.

5. Results

The sufficient conditions for comparing the extremal indexes of stationary waiting time in the $M/G/1$ and $GI/M/1$ systems are obtained. We have proven that if in both systems the equation $\mathrm{e}e^{\gamma(S-\tau)} = 1$ has a real positive roots and the interarrival intervals and the service times are ordered in failure rate then the extremal indexes are ordered. For $M/G/1$ the obtained result is extended to the case of a system with mixed service times with ordered components. In the case of multiserver $GI/M/c$ system we have shown that it is also possible to establish a comparison of the extremal indexes. We illustrate this results on examples with some special distributions. For $GI/G/1$ system with service time determined by a finite mixture whose dominant component of the equilibrium distribution belongs to the class of subexponential distributions,we have proven that the tail of the limiting distribution of the stationary waiting time is equivalent to the tail of this distribution up to a constant, the form of which is obtained. Furthermore, the limiting distribution of the maximum of the stationary waiting time belongs to the maximum domain of attraction of the distribution of extreme values of the same type as the maximum of the random variables defined by the dominant component, while the extremal index of waiting time is zero.

6. Discussion

The numerical examples given in Sections 2 and 3 of the article demonstrate the correctness of the obtained statements. Note that applying the order in failure rate makes it possible to compare systems with different interarrival distributions and different service time distributions (not just with identical distributions but different parameters).

To demonstrate the assertion of Theorem 5, we considered a system with an exponential-Pareto service-time distribution. The results of our numerical experiments show that the asymptotic distribution of maximum waiting time works well when the traffic is light. Continuing the present work, we plan to extend the numerical experiments with different distributions and investigate the sensitivity of the approximation scheme to the group size n and the traffic intensity ρ.

7. Conclusion

The research examined the conditions under which the extremal indexes of two queuing systems can be compared. This can be used to select parameters for systems that guarantee a given value of the extremal index. The extreme behaviour of the stationary waiting time is also considered.

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Экстремальный индекс времени ожидания в системе GI/G/1

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Аннотация. В данной работе получены условия сравнения экстремального индекса стационарного времени ожидания в системах $M/G/1$ и $GI/M/1$. Эти условия включают экспоненциальное асимптотическое поведение хвоста времени ожидания и порядок по интенсивности отказов для интервалов между приходами заявок и для времени обслуживания в сравниваемых системах. Для системы $M/G/1$ полученный результат распространяется на смешанные времена обслуживания с упорядоченными компонентами. Если в системе $GI/G/1$ время обслуживания определяется конечной смесью, доминирующая компонента равновесного распределения которой принадлежит классу субэкспоненциальных распределений, то хвост предельного распределения стационарного времени ожидания эквивалентен хвосту этого распределения с точностью до константы, вычисленной в явном виде. Кроме того, предельное распределение максимума стационарного времени ожидания принадлежит области максимального притяжения распределения экстремальных значений того же типа, что и максимум случайных величин, определяемых доминирующей компонентой.

Ключевые слова: экстремальный индекс, система обслуживания, упорядоченность по интенсивности отказа