Discrete & Continuous Models
& Applied Computational Science
ISSN 2658-7149 (Online), 2658-4670 (Print)

2025, 33 (4) 404–410
http://journals.rudn.ru/miph

Research article

UDC 519.872, 519.217
PACS 07.05.Tp, 02.60.Pn, 02.70.Bf
DOI: 10.22363/2658-4670-2025-33-4-404-410
EDN: HSZAJF

On calculating the dimension of invariant sets of dynamic systems

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(received: October 12, 2025; revised: October 25, 2025; accepted: October 30, 2025)

Abstract. This work investigates numerical approaches for estimating the dimension of invariant sets onto which the trajectories of dynamic systems “wind”, with a focus on fractal and correlation dimensions. While the classical fractal dimension becomes computationally challenging in spaces of dimension greater than two, the correlation dimension offers a more efficient and scalable alternative. We develop and implement a computational method for evaluating the correlation dimension of large discrete point sets generated by numerical integration of differential equations. An analogy is noted between this approach and the Richardson–Kalitkin method for estimating the error of a numerical method. The method is tested on two representative systems: a conservative system whose orbit lies on a two-dimensional torus, and the Lorenz system, a canonical example of a chaotic flow with a non-integer attractor dimension. In both cases, the estimated correlation dimensions agree with theoretical predictions and previously reported results. The developed software provides an effective tool for analysing invariant manifolds of dynamical systems and is suitable for further studies, including those involving reversible difference schemes and high-dimensional systems.

Key words and phrases: correlation dimension, dynamic systems

For citation: Kadrov, V. M., Malykh, M. D. On calculating the dimension of invariant sets of dynamic systems. *Discrete and Continuous Models and Applied Computational Science* **33** (4), 404–410. doi: 10.22363/2658-4670-2025-33-4-404-410. edn: HSZAJF (2025).

1. Introduction

Numerical methods for integrating dynamic systems make it possible to find many thousands and even millions of points along a system’s trajectory. However, the invariant sets that arise along the way do not always look like curved lines [1], so the challenge arises of developing software for calculating the dimensions of such invariant “manifolds.”

Taking the neighborhood of a point on a smooth curve or surface, we expect to see how the curve turns into a line, and the surface into a plane, as this neighborhood is reduced. However, as early as the 19th century, the first examples of functions whose graphs do not become simpler with increasing scale appeared. Classic examples of such lines are the graph of the Weierstrass function or the trajectory of Brownian motion. A quantitative measure characterizing such an irregular structure was introduced in the 1980s and is called the *fractal dimension* [2].

Although the use of fractal dimension for studying dynamic systems with deterministic chaos [3, 4] is inferior in many ways to the methods of qualitative analysis of such systems [5], the method itself has found wide application in assessing the complexity and predictability of physical signals in natural systems, primarily biological ones [6], as well as in studying the interactions of coupled dynamic systems [7, 8]. We will consider the calculation of dimension as a method that, with relatively low computational costs, can provide a description of the invariant set of a dynamic system.

2. Fractal dimension

The classical approach to determining the fractal dimension of a set (see [2, 9]) is as follows. We describe an n-dimensional grid with a step of r around the set M under study. Let $N(r)$ be the number of cubes with sides r required to cover the set M. Then, as the linear dimensions of the grid decrease, the number of cubes asymptotically behaves as

$$N(r) \sim r^D, \quad (r \to 0),$$

where D is the fractal dimension of the set.

Although this method is quite intuitive, calculations for systems with dimensions greater than two present some difficulties. It should also be noted that it does not take into account the frequency of trajectories hitting a specific cube [10].

3. Correlation dimension

As an alternative to the fractal dimension, similar constructs can be used, among which we have chosen the correlation dimension, originally proposed for time series analysis [9–11].

Let N be a sufficiently large natural number, and $\{\vec{x}_1, \dots, \vec{x}_N\}$ be an arbitrary subset of points in the set M. The correlation sum is defined as

$$C(r) = \frac{1}{N^2} \sum_{i,j} [d(\vec{x}_i, \vec{x}_j) < r],$$

where $d(\vec{x}, \vec{y})$ is the Euclidean distance between points, and $\sum_{i,j} [d(\vec{x}_i, \vec{x}_j) < r]$ is the number of pairs of points satisfying the condition $d(\vec{x}_i, \vec{x}_j) < r$. If, for sufficiently small r, the correlation sum behaves as a power of r, that is,

$$C(r) \simeq r^{D_c}, \tag{1}$$

then its exponent D_c is called the correlation dimension of the set M. In Ref. [10] it is proved that D_c is a lower bound for the fractal dimension.

Defining the dimension of a set as a correlation dimension is computationally more convenient than using a fractal dimension, since the number of operations depends only on the number of points. Not less important is that the correlation dimension can also be applied to manifolds embedded in a high-dimensional space.

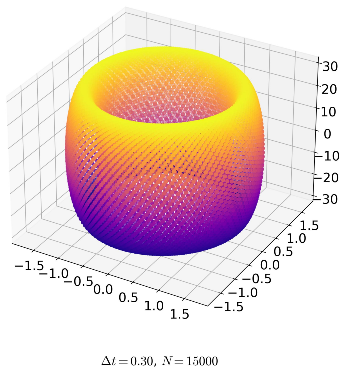

Figure 1. Solution of the system (2) from Example 1

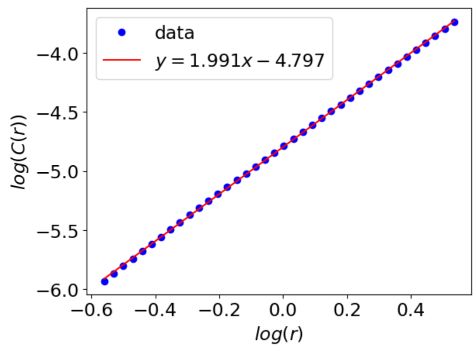

Figure 2. Dependence of the correlation sum on the radius r of the solution of the system (2) from Example 1

4. Calculation of correlation dimension

The calculation of correlation dimension of a given set of points was carried out using a method largely similar to the Richardson–Kalitkin method [12–16]. For various values of parameter r, the values of C were calculated and the dependence of C on r was plotted in logarithmic scale. Then the slope, which is interpreted as the correlation dimension, was determined using the least squares method.

When choosing the range for the parameter r, it should be noted that there are:

1) the maximum value r_{max}, above which all points of the finite sequence of points in the set M will be included in the correlation sum,
2) the minimum value r_{min}, below which the correlation sum is zero.

In practice, the linear segment can be much smaller than $[r_{min}, r_{max}]$, and the estimate can be corrupted by "tails" where the relation (1) is not satisfied. As in Richardson's method, these tails on the left, for small r, are due to roundoff error, while those on the right are due to the role of terms omitted in (1).

If the points $\{\vec{x}_i\}$ fill the classical manifold more or less uniformly, and the number N is sufficiently large, then the correlation dimension coincides with the ordinary dimension of the manifold. We will interpret the presence of a linear section on the logarithmic plot of C versus r as a condition for the applicability of the method.

5. Interpretation of the obtained value

When numerically integrating dynamical systems, we obtain a finite set of orbital points. Using this set of points as the set $\{\vec{x}_i\}$, we can calculate the correlation dimension of the orbit of a continuous dynamical system by making a number of difficult-to-formalize assumptions. The results of such calculations often coincide with theoretical predictions or, at least, are close to those obtained by other methods [11].

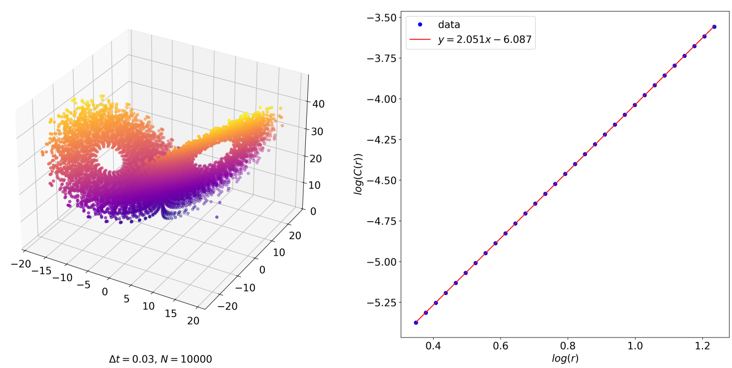

Figure 3. Solution of the Lorenz system (Example 2) and a plot of the correlation sum versus the radius r

Example 1. Consider a conservative system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + 2\epsilon yz, \\ \dot{z} = 1 - y^2 - \epsilon z^2. \end{cases} \tag{2}$$

For a small perturbation $\epsilon = 10^{-4}$, the trajectory passes along a two-dimensional torus [5, p. 3.4.1], which is clearly seen in Fig. 1, obtained by numerically integrating this system at $(x_0, y_0, z_0) = (1.7, 0, 0)$. The correlation dimension D_c of this set will be the slope of the linear section of Fig. 2, constructed on a double logarithmic scale. Therefore, in this way, the dimension is estimated at $D_C \approx 1.99$ against the geometric dimension $D = 2$.

Example 2. A well-known example of a system whose trajectory, found approximately by the Runge–Kutta method, has a non-integer dimension is E. Lorenz's system of convection equations.

$$\begin{cases} \dot{x} = 10(y - x), \\ \dot{y} = x(28 - z) - y, \\ \dot{z} = xy - \dfrac{8}{3}z. \end{cases}$$

Its dimension $D \approx 2.06$ was calculated by various methods [17, 18], and the value of D_c is also close to this number (Fig. 3).

6. Results

The method for computing the correlation dimension enables the efficient estimation of a set of points — such as, in our case, the trajectory trace of a dynamical system. As the examples demonstrate, the method is capable of accurately determining the dimensionality of both regular sets and those commonly classified as fractal. For a conservative system whose trajectory lies on a two-dimensional torus, we obtained a value close to 2; similarly, the estimated dimension of the Lorenz attractor is consistent with values reported in other studies. In contrast to the fractal dimension, the correlation-dimension approach readily allows the computation of this characteristic even for sets embedded in spaces of substantially higher dimension than three.

7. Discussion

In the numerical study of an integral curve, two parameters arise that characterize the numerical method: the time step Δt and the number of points N considered. Their product $T = N\Delta t$ yields the time interval under consideration. In Ref. [11], it was noted that these parameters significantly influence the dimensionality estimate.

If T is small, then what is actually being calculated is the germ of the integral curve, whose points fit perfectly onto the one-dimensional analytic curve by virtue of the local Cauchy theorem. Of course, we do not know the characteristic time T for a dynamical system, during which the system begins to exhibit its "global" properties. If Δt is not small enough, then the numerical solution may differ significantly from the exact solution, and therefore the calculated dimensionality will have no relation to the dimensionality of the desired orbit. Excessive reduction of Δt for a fixed T is very difficult, since it leads to prohibitively complex calculations. Despite the existence of theoretical estimates for the minimum number of points [19, 20], we will adhere to the approach of Ref. [11], which considered four chaotic systems and showed that 6,000-7,000 points were needed to obtain all the information about the trajectory, which is not very many by modern standards.

Some of these issues are resolved by considering the difference scheme as a discrete model describing the same physical phenomenon no worse than a continuous one. With this approach, the question of the dimensionality of the system's orbital closure is separated from the question of how this dimensionality changes depending on the step size Δt. Continuing the present work, we plan to use the software presented here to investigate the dimensionality of invariant sets found using reversible schemes with fixed and even deliberately large time steps [1].

8. Conclusion

The study examined approaches for estimating the dimension of a discrete set, specifically the fractal and correlation dimensions. The latter is more suitable for computational analyses in the task of determining the dimension of an invariant set of dynamical systems; therefore, experiments were conducted using two representative examples: a system whose trajectory lies on a two-dimensional torus, serving as an example of a regular set, and the Lorenz system as an extreme case. In both instances, the method produced satisfactory dimension estimates. The developed program for computing the dimension will facilitate future investigations of invariant manifolds of dynamical systems, particularly in combination with reversible difference schemes.

Author Contributions: Conceptualization, M. Malykh; methodology, M. Malykh and V. Kadrov; software, V. Kadrov; validation, V. Kadrov; writing—original draft preparation, V. Kadrov; writing—review and editing, M. Malykh. All authors have read and agreed to the published version of the manuscript.

Funding: The work was carried out with the financial support of the Russian Science Foundation (project No. 20-11-20257).

Data Availability Statement: Data sharing is not applicable.

Acknowledgments: The authors are grateful to prof. L.A. Sevastianov (RUDN) for attention to their research.

Conflicts of Interest: The authors declare no conflict of interest.

Declaration on Generative AI: The authors have not employed any Generative AI tools.

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УДК 519.872, 519.217
PACS 07.05.Tp, 02.60.Pn, 02.70.Bf
DOI: 10.22363/2658-4670-2025-33-4-404-410 EDN: HSZAJF

О вычислении размерности инвариантных множеств динамических систем

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Аннотация. В работе рассматриваются численные подходы к оценке размерности инвариантных множеств, на которые «навиваются» траектории динамических систем: методы расчёта фрактальной и корреляционной размерности. Классическая фрактальная размерность становится вычислительно трудоёмкой при работе с пространствами размерности выше двух, тогда как корреляционная размерность представляет собой более эффективную альтернативу. Разработан и реализован вычислительный метод для оценки корреляционной размерности больших дискретных наборов точек, полученных в результате численного интегрирования дифференциальных уравнений. Отмечена аналогия данного подхода с методом Ричардсона–Калиткина для оценки погрешности численного метода. Предложенный метод протестирован на двух характерных примерах: консервативной системе, чья орбита лежит на двумерном торе, и системе Лоренца — классическом примере хаотической система с нецелой размерностью аттрактора. В обоих случаях полученные оценки корреляционной размерности согласуются с теорией и ранее опубликованными результатами. Разработанное программное обеспечение послужит эффективным инструментом для анализа инвариантных многообразий динамических систем и подходит для дальнейших исследований, в особенности для компьютерных экспериментов с использованием обратимых разностных схем, а также для систем высокой размерности.

Ключевые слова: корреляционная размерность, динамические системы