Discrete & Continuous Models
& Applied Computational Science
ISSN 2658-7149 (Online), 2658-4670 (Print)

2024, 32 (4) 425–444
http://journals.rudn.ru/miph

Research article

UDC 519.872:519.217
PACS 07.05.Tp, 02.60.Pn, 02.70.Bf
DOI: 10.22363/2658-4670-2024-32-4-425-444
EDN: DIOZWP

Development and adaptation of higher-order iterative methods in R^n with specific rules

T. Zhanlav[1,2], Kh. Otgondorj[2]

[1] Institute of Mathematics and Digital Technology, Mongolian Academy of Sciences, Ulaanbator, 13330, Mongolia

[2] Mongolian University of Science and Technology, Ulaanbator, 14191, Mongolia

(received: September 21, 2024; revised: October 5, 2024; accepted: October 6, 2024)

Abstract. In this article, we propose fourth- and fifth-order two-step iterative methods for solving the systems of nonlinear equations in R^n with the operations of multiplication and division of vectors. Some of the proposed optimal fourth-order methods are considered as an extension of well-known methods that designed only for solving the nonlinear equations. We also developed $p\,(5 \leqslant p \leqslant 8)$—order three-point iterative methods for solving the systems of nonlinear equations, that contain some known iterations as particular cases. The computational efficiency of the new methods has been calculated and compared. The outcomes of numerical experiments are given to support the theoretical results concerning convergence order and computational efficiency. Comparative analysis demonstrates the superiority of the developed numerical techniques.

Key words and phrases: nonlinear systems, newton-type methods, order of convergence, computational efficiency, three-step iteration

For citation: Zhanlav, T., Otgondorj, K. Development and adaptation of higher-order iterative methods in R^n with specific rules. *Discrete and Continuous Models and Applied Computational Science* **32** (4), 425–444. doi: 10.22363/2658-4670-2024-32-4-425-444. edn: DIOZWP (2024).

1. Introduction

The problem to find a real solution of nonlinear system

$$F(x) = 0, \quad x = (x_1, x_2, \dots, x_n)^T \in R^n \tag{1}$$

has many applications in sciences and engineering [1–19]. In general, the root (zero) of equation (1) cannot be computed exactly. Most of the numerical methods used for solving this problem are iterative ones. Recently, many high-order iterative methods are presented in literature, see [1–9, 14, 17, 19] and references therein. Some methods [7, 8] of multiplication and division of two vectors, understood component-wise, are used. Let $a = (a_1, a_2, \dots, a_n)^T \in R^n$ and $b = (b_1, b_2, \dots, b_n)^T \in R^n$. Then

$$a \cdot b = (a_1 b_1, a_2 b_2, \dots, a_n b_n)^T \in R^n, \tag{2}$$

$$\frac{a}{b} = \left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}\right)^T \in R^n. \tag{3}$$

The direct consequence of (2) is

$$a \cdot b = b \cdot a. \tag{4}$$

If $a = b$ then (2) and (3) can be written as

$$a^2 = a \cdot a = \left(a_1^2, a_2^2, \dots, a_n^2\right)^T \in R^n,$$
$$\mathbf{1} = \left(1, 1, \dots, 1\right)^T \in R^n.$$

The purpose of this paper is to construct higher-order iterative methods in R^n with multiplication and division rules (2) and (3). For obtaining the numerical solutions of the equation (1) we often use the two-step and three-step iterative methods as shown below:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= y_k - \bar{T}_k F'(x_k)^{-1}F(y_k) \text{ or } x_{k+1} = x_k - T_k F'(x_k)^{-1}F(x_k), \end{aligned} \tag{5}$$

and

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ z_k &= y_k - \bar{T}_k F'(x_k)^{-1}F(y_k), \\ x_{k+1} &= z_k - \bar{\Psi}_k F'(x_k)^{-1}F(z_k). \end{aligned} \tag{6}$$

Here $\bar{T}_k$ and $\bar{\Psi}_k$ are iteration parameters to be determined properly. The convergence order of iterations (5) and (6) we denote by p and $\rho = p + q$ respectively. We use C''- convergence order [2] based on nonlinear residual:

$$\frac{\|F(x_{k+1})\|}{\|F(x_k)\|^\sigma} = \text{const}$$

or

$$F(x_{k+1}) = O(h^\sigma), \quad h = F(x_k).$$

In our previous papers [11, 14] we find the sufficient convergence conditions in term of parameters $\bar{T}_k$ and $\bar{\Psi}_k$:

$$\bar{T}_k = I + O(h), \tag{7a}$$
$$\bar{T}_k = I + 2\Theta_k + O(h^2), \tag{7b}$$
$$\bar{T}_k = I + 2\Theta_k + 3d_k + 5\Theta_k^2 + O(h^3), \tag{7c}$$

and

$$\bar{\Psi}_k = I + O(h), \tag{8a}$$
$$\bar{\Psi}_k = I + 2\Theta_k + O(h^2), \tag{8b}$$
$$\bar{\Psi}_k = I + 2\Theta_k + 3d_k + 6\Theta_k^2 + O(h^3), \tag{8c}$$

where

$$\begin{aligned} \Theta_k &= \frac{1}{2}F'(x_k)^{-1}F''(x_k)\xi_k, \\ d_k &= -\frac{1}{6}F'(x_k)^{-1}F'''(x_k)\xi_k^2, \\ \xi_k &= F'(x_k)^{-1}F(x_k). \end{aligned} \tag{9}$$

Table 1
Summarizing the results of [11]

p	$\bar{T}_k$	q	$\bar{\Psi}_k$
3	(7a)	1	(8a)
4	(7b)	2	(8b)
5	(7c)	3	(8c)

Table 2
Summarizing the results of [14]

$\rho = p + q$	$\bar{T}_k$	$\bar{\Psi}_k$
5	(7a)	(8b)
	(7b)	(8a)
6	(7b)	(8b)
	(7c)	(8a)
	(7a)	(8c)
7	(7b)	(8c)
	(7c)	(8b)
8	(7c)	(8c)

Summarizing the results of [11, 14], we present in Tables 1, 2 the convergence orders p and ρ of iterations (5) and (6) respectively.

It is worth noting that the convergence of iterations (5) and (6) was proved under the following condition

$$\frac{1}{2}F'(x_k)^{-1}F''(x_k)\Theta_k F'(x_k)^{-1}F(x_k) = \Theta_k^2 + O(h^3), \tag{10}$$

which holds true due to permutation properties of q-derivatives ($q \geq 1$) [20]. This paper is organized in five sections. Section 2 is devoted to constructing fourth-order two-step iterations and family of parametric two-step iterations. Extensions of some well known scalar methods with fourth order of convergence to solve systems of nonlinear equations are also discussed in this section. In Section 3, new two-parametric family of sixth, seventh, and eighth-order three-point iterative methods are constructed. Computational efficiency of the developed iterations is discussed in Section 4. In Section 5, we describe the outcomes of numerical experiments to confirm the theoretical analysis and made a comparison of some methods. Finally, we close with conclusions.

2. Construction of new iterative methods

First, we consider the following two-step iterations:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= x_k - \Omega_k, \end{aligned} \tag{11}$$

where

$$\Omega_k = \left(aF'(x_k) + bF'(\eta_k) + cF'(y_k)\right)^{-1}\left(\alpha F(y_k) + \beta F(x_k) + \gamma F(w_k)\right), \tag{12}$$

$w_k = x_k + \xi_k$, $\eta_k = x_k - \frac{2}{3}\xi_k$ and $a + b + c \neq 0$, a, b, c, α, β, γ are real constants. The convergence of our iteration (11) is established.

Theorem 5. *The convergence order of iteration* (11) *is equal to four, iff*

$$\begin{aligned} &b = \frac{3}{2}(1-2\gamma)(1-4\gamma), \quad a = 1 - 2\gamma - b, \quad c = 2\gamma, \\ &\beta = 1 - 2\gamma, \quad \alpha = 1 - 5\gamma - \frac{4}{3}b. \end{aligned} \tag{13}$$

Proof. The Taylor expansions of $F'(y_k)$ and $F'(\eta_k)$ at point x_k give

$$F'(y_k) = F'(x_k)(I - 2\Theta_k - 3d_k) + O(h^3),$$
$$F'(\eta_k) = F'(x_k)(I - \frac{4}{3}(\Theta_k + d_k) + O(h^3).$$

Hence, we have

$$aF'(x_k) + bF'(\eta_k) + cF'(y_k) = F'(x_k)\Big((a+b+c)I - \Big(\frac{4}{3}b(\Theta_k + d_k) + c(2\Theta_k + 3d_k)\Big)\Big) + O(h^3).$$

From this we get

$$\big(aF'(x_k) + bF'(\eta_k) + cF'(y_k)\big)^{-1} = \frac{1}{a+b+c}\Big(I + \frac{1}{a+b+c}\Big(\Big(2c + \frac{4}{3}b\Big)\Theta_k + \\ + \Big(\frac{4}{3}b + 3c\Big)d_k\Big) + \frac{1}{(a+b+c)^2}\Big(\frac{16}{9}b^2 + 4c + \frac{16}{3}bc\Big)\Theta_k^2\Big)F'(x_k)^{-1} + O(h^3).$$

Similarly, using Taylor expansions of $F(y_k)$ and $F(w_k)$, we get

$$F'(x_k)^{-1}\big(\alpha F(y_k) + \beta F(x_k) + \gamma F(w_k)\big) = \Big((\beta + 2\gamma)I + (\alpha + \gamma)\Theta_k + (\alpha - \gamma)d_k\Big)\xi_k + O(h^4).$$

Hence, Ω_k given by (12) can be rewritten as

$$\Omega_k = \frac{1}{a+b+c}\Big((\beta + 2\gamma)I + \Big(\frac{\beta + 2\gamma}{a+b+c}(2c + \frac{4}{3}b) + \alpha + \gamma\Big)\Theta_k + \Big(\alpha - \gamma + \frac{\beta + 2\gamma}{a+b+c}(3c + \frac{4}{3}b)\Big)d_k + \\ + \Big(\frac{\beta + 2\gamma}{(a+b+c)^2}(\frac{16}{9}b^2 + 4c + \frac{16}{3}bc) + \frac{\alpha + \gamma}{a+b+c}(2c + \frac{4}{3}b)\Big)\Theta_k^2\Big)\xi_k + O(h^3).$$

We find the unknown coefficients in (11) such that [11, 14]

$$\Omega_k = (I + \Theta_k + d_k + 2\Theta_k^2)\xi_k.$$

This condition gives us

$$\begin{aligned} &a + b + c = 1, \quad \beta + 2\gamma = 1, \quad 2c + \frac{4}{3}b + \alpha + \gamma = 1, \\ &\alpha - \gamma + 3c + \frac{4}{3}b = 1, \quad \frac{16}{9}b^2 + 4c + \frac{16}{3}bc + (\alpha + \gamma)\big(2c + \frac{4}{3}b\big) = 2. \end{aligned} \tag{14}$$

The solution of system (14) is (13). □

Thus, the iteration (11) becomes as:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= x_k - \Omega_k(\gamma), \end{aligned} \tag{15}$$

where

$$\Omega_k(\gamma) = \Big((1 - 2\gamma - b)F'(x_k) + bF'(\eta_k) + 2\gamma F'(y_k)\Big)^{-1}\Big((1 - 5\gamma - \frac{4}{3}b)F(y_k) \\ + (1 - 2\gamma)F(x_k) + \gamma F(w_k)\Big), \quad b = \frac{3}{2}(1 - 2\gamma)(1 - 4\gamma),$$

$$\bar{T}_k = \frac{\Omega_k(\gamma)}{\Theta_k \xi_k}. \tag{16}$$

This family includes some well known methods as particular cases. We consider the particular cases of family (15):

1. Let $\gamma = \frac{1}{2}$. Then by (13) we get $a = b = \beta = 0$, $c = 1$, $\alpha = -\frac{3}{2}$ and (15) leads to

$$y_k = x_k - F'(x_k)^{-1}F(x_k),$$
$$x_{k+1} = x_k - F'(y_k)^{-1}\Big(-\frac{3}{2}F(y_k) + \frac{1}{2}F(w_k)\Big),$$

which is the first iteration obtained by Su in [9].

2. Let $\gamma = 0$. Then by (13) we get $\beta = 1$, $c = 0$, $b = \frac{3}{2}$, $a = -\frac{1}{2}$, $\alpha = -1$ and (15) leads to

$$y_k = x_k - F'(x_k)^{-1}F(x_k),$$
$$x_{k+1} = x_k - \Big(\frac{1}{2}F'(x_k) - \frac{3}{2}F'(y_k)\Big)^{-1}\Big(F(y_k) - F(x_k)\Big),$$

which is the second iteration obtained by Su in [9].

3. Let $\gamma = \frac{1}{4}$. Then by (13) we get $a = c = \beta = \frac{1}{2}$, $b = 0$, $\alpha = -\frac{1}{4}$ and (15) leads to

$$y_k = x_k - F'(x_k)^{-1}F(x_k),$$
$$x_{k+1} = x_k - \Big(F'(x_k) + F'(y_k)\Big)^{-1}\Big(F(x_k) + \frac{1}{2}F(w_k) - \frac{1}{2}F(y_k)\Big).$$

The following lemma plays key role in constructing high-order iterations.

Lemma 1. *The Θ_k given by* (9) *is equal to*

$$\Theta_k = \frac{F(y_k)}{F(x_k)} + O(h^2). \tag{17}$$

Proof. The Taylor expansion of $F(y_k)$ at point x_k gives

$$F(y_k) = \frac{1}{2}F''(x_k)\xi_k^2 - \frac{1}{6}F'''(x_k)\xi_k^3 + O(h^4).$$

Then

$$F(x_k)^{-1}F(y_k) = (\Theta_k + d_k)\xi_k = \Theta_k\xi_k + O(h^3),$$

and thereby using (10) we obtain

$$\begin{aligned} F'(x_k)^{-1}F''(x_k)F'(x_k)^{-1}F(y_k) &= F'(x_k)^{-1}F''(x_k)\Theta_k\xi_k + O(h^3) \\ &= 2\Theta_k^2 + O(h^3). \end{aligned} \tag{18}$$

On the other hand, the left-hand side of (18) can be described as:

$$F'(x_k)^{-1}F''(x_k)F'(x_k)^{-1}F(x_k)\frac{F(y_k)}{F(x_k)} = 2\Theta_k\frac{F(y_k)}{F(x_k)}. \tag{19}$$

From (18) and (19) we reach (17). □

Note that the same definition (17) for Θ_k was used in the scalar equation case [11]. In $\mathbb{R}^n$ with operations (2) and (3) the iteration parameters $\bar{T}_k$ and $\bar{\Psi}_k$ turn out to be determined as vectors, that permit essentially simplification of implementation algorithms as compared with other existing methods with same order of convergence.

Theorem 6. *The two-step iteration* (5) *has a third, fourth and fifth-order convergence if and only if the parameter* $\bar{T}_k$ *satisfies*

$$\bar{T}_k = \mathbf{1} + O(h), \tag{20a}$$

$$\bar{T}_k = \mathbf{1} + 2\Theta_k + O(h^2), \tag{20b}$$

$$\bar{T}_k F'(x_k)^{-1}F(y_k) = \{\mathbf{1} + \Theta_k^2\}F'(y_k)^{-1}F(y_k), \tag{20c}$$

where $\Theta_k = \dfrac{F(y_k)}{F(x_k)}$, $\mathbf{1} = (1, 1, \dots, 1)^T \in \mathbb{R}^n$.

Proof. In [21] was shown that the fourth order convergence condition [12] is equivalent to:

$$\bar{T}_k = \mathbf{1} + \Theta_k + 2\Theta_k^2 + O(h^3). \tag{21}$$

Using (17), (21) and

$$\bar{T}_k = \frac{T_k - I}{\Theta_k},$$

we obtain

$$\bar{T}_k = \mathbf{1} + 2\Theta_k + O(h^2).$$

Analogously, using (20c) we obtain $F(x_{k+1}) = O(h^5)$. □

Using the expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots, \quad \|x\| = \max_i |x_i| < 1, \quad x \in \mathbb{R}^n,$$

it is easy to show that $\bar{T}_k$ given by formula

$$\bar{T}_k = \frac{\mathbf{1} + a\Theta_k + b\Theta_k^2}{\mathbf{1} + (a-2)\Theta_k + d\Theta_k^2}, \quad a,\, b,\, d \in \mathbb{R}, \quad \Theta_k = \frac{F(y_k)}{F(x_k)}, \tag{22}$$

satisfies the conditions (20a) and (20b). In this case, the two-step iteration with (22) can be written as:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= y_k - \frac{F(x_k)^2 + aF(x_k)F(y_k) + bF(y_k)^2}{F(x_k)^2 + (a-2)F(x_k)F(y_k) + dF(y_k)^2} F'(x_k)^{-1}F(y_k), \quad a,\, b,\, d \in \mathbb{R}. \end{aligned} \tag{23}$$

Thus, we have obtained another family of fourth-order iterations (23). Now we consider some particular cases of this family.

1. Let $b = d = 0$. Then (23) leads to

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= y_k - \frac{\mathbf{1} + a\Theta_k}{\mathbf{1} + (a-2)\Theta_k} F'(x_k)^{-1}F(y_k), \quad a \in \mathbb{R}. \end{aligned}$$

   This is a generalization of King's method for the system (1).
2. Let $a = b = 0$, $d = 1$. Then (23) leads to

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= y_k - \frac{\mathbf{1}}{(\mathbf{1} - \Theta_k)^2} F'(x_k)^{-1}F(y_k), \quad a \in \mathbb{R}, \end{aligned}$$

   which is a generalization of King and Traub's method for the system (1).

3. Let $a = 1$, $b = -1$, $d = 0$. Then (23) leads to

$$y_k = x_k - F'(x_k)^{-1}F(x_k),$$
$$x_{k+1} = y_k - \frac{\mathbf{1} + \Theta_k - \Theta_k^2}{\mathbf{1} - \Theta_k} F'(x_k)^{-1}F(y_k),$$

which is a generalization of Maheshwari's method for the system (1).

Note that similar extension of King's, King and Traub's and Chun's methods are suggested in [20]. According to the definition given in [20], the family of iteration (23) is the optimal fourth-order one. Similarly, one can construct the generalization of all known fourth-order methods for system (1). Analogously, the following theorem is proved:

Theorem 7. *The convergence order of two-step iterations* (5) *equal to five, if $\bar{T}_k$ satisfies*

$$\bar{T}_k = (dI + \alpha t_k + \beta t_k^2)^{-1}(aI + bt_k + ct_k^2), \tag{24}$$

where

$$\begin{aligned} a &= \frac{(3\alpha - 26\beta + 13c)}{5}, \quad b = \frac{(\alpha + 18\beta - 14c)}{5} \\ d &= \frac{(4c - 13\beta - \alpha)}{5}, \quad \alpha + \beta + c \neq 0, \quad \alpha,\ \beta,\ c \in R. \end{aligned} \tag{25}$$

Proof. Using the formula (18) and

$$t_k = F'(x_k)^{-1}F'(y_k) = I - 2\Theta_k - 3d_k + O(h^3), \quad s_k = F'(y_k)^{-1}F'(x_k),$$

one can easily shown that (24) satisfies (7c) under (25). □

We consider some special case of (24), (25).

1. Let $c = \beta = 0$. Then by (25) we get $a = \frac{3\alpha}{5}$, $b = -\frac{\alpha}{5}$, $d = \frac{3\alpha}{5}$. Substituting these values into (24) we obtain

$$\bar{T}_k = (5t_k - I)^{-1}(3I + t_k),$$

which is obtained by Wang in [6]. Note that his result is a generalization of method Ham and Chun (HC5) constructed for the scalar equation case [5].

2. Let $\alpha = \beta = 0$. Then by (25) we get $a = \frac{13c}{5}$, $b = -\frac{14c}{5}$, $d = \frac{4c}{5}$. Substituting these values into (24) we obtain

$$\bar{T}_k = \frac{13}{4}I - \frac{7}{2}t_k + \frac{5}{4}t_k^2,$$

which coincides with result [14, 18].

The formula (24) can be rewritten also in term of s_k as:

$$\bar{T}_k = (\beta I + \alpha s_k + ds_k^2)^{-1}(cI + bs_k + as_k^2),$$

which includes choices of Cordero et. al. [1] as particular cases.

Now we consider the following two-step iterations

$$\begin{aligned} y_k &= x_k - \bar{a}F'(x_k)^{-1}F(x_k), \quad \bar{a} \in R, \quad \bar{a} \neq 0, \\ z_k &= y_k - \bar{T}_k F'(x_k)^{-1}F(y_k). \end{aligned} \tag{26}$$

For iteration (26), the following result holds:

Theorem 8. *The convergence order of the family of iterations* (26) *equal to four (five, when* $\bar{a} = 1$ *) if* $\bar{T}_k$ *is given by*

$$\bar{T}_k = \mathbf{1} + (\bar{a}+1)\bar{\Theta}_k + (\bar{a}^2 + 2\bar{a} + 2)\bar{\Theta}_k^2 + (\bar{a}^2 + \bar{a} + 1)d_k, \tag{27}$$

where

$$\bar{\Theta}_k = \frac{1}{\bar{a}^2}\left(\frac{F(y_k) + F(w_k)}{2F(x_k)} - \mathbf{1}\right), \tag{28}$$

$$d_k = \frac{1}{\bar{a}^2}\left(\frac{F(y_k) - F(w_k)}{2\bar{a}F(x_k)} + \mathbf{1}\right), \tag{29}$$

$$w_k = x_k + \bar{a}F'(x_k)^{-1}F(x_k).$$

Proof. As in proof of lemma (1), it is easy to show that

$$\begin{aligned} \frac{F(y_k)}{F(x_k)} &= (1-\bar{a})\mathbf{1} + \bar{a}^2\bar{\Theta}_k + \bar{a}^3 d_k + O(h^3), \\ \frac{F(w_k)}{F(x_k)} &= (1+\bar{a})\mathbf{1} + \bar{a}^2\bar{\Theta}_k - \bar{a}^3 d_k + O(h^3). \end{aligned} \tag{30}$$

From (30) we find $\bar{\Theta}_k$ and d_k and obtain (28), (29) with accuracy $O(h^3)$. So fourth order (fifth order when $\bar{a} = 1$) convergence condition [14] is satisfied for (27) with (28), (29). □

The adaptation of formula (27) in R^n with operations of multiplication and division of vectors extremely easily realized by (28), (29).

Note that in (26) only one inverse of $F'(x_k)$ is required, whereas in the iteration (5) with $\bar{T}_k$ defined by (20c) two inverses of $F'(x_k)$ and $F'(y_k)$ are required.

3. The three-step methods

Now consider three-step iterations

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ z_k &= \phi_k(x_k, y_k), \\ x_{k+1} &= z_k - \bar{\Psi}_k F'(x_k)^{-1}F(z_k). \end{aligned} \tag{31}$$

Here $z_k = \phi_k(x_k, y_k)$ is the iteration function of order $p \geq 2$.

Theorem 9. *The methods* (31) *have order of convergence* $p+1$, $p+2$, $p+3$ *if and only if the parameter* $\bar{\Psi}_k$ *satisfies*

$$\bar{\Psi}_k = \mathbf{1} + O(h), \tag{32a}$$

$$\bar{\Psi}_k = \mathbf{1} + 2\Theta_k + O(h^2), \tag{32b}$$

$$\bar{\Psi}_k F'(x_k)^{-1}F(z_k) = \{\mathbf{1} + 2\Theta_k^2\}F'(y_k)^{-1}F(z_k), \tag{32c}$$

respectively. The proof of this theorem is the same as Theorem 7 thus we omit it here. By virtue of Theorem 2 in [14] the iterations (31) has order of convergence $p+3$ if and only if $\bar{\Psi}_k$ satisfies (see also Table 1).

$$\bar{\Psi}_k = I + 2\Theta_k + 3d_k + 6\Theta_k^2 + O(h^3). \tag{33}$$

Using Taylor expansions of $F'(y_k)$ and $F'(y_k)^{-1}$ one can easily show that (33) equivalent to:

$$\bar{\Psi}_k = F'(y_k)^{-1}F'(x_k) + 2\Theta_k^2 + O(h^3), \quad \Theta_k = \frac{F(y_k)}{F(x_k)}.$$

Then by (4) we get

$$\begin{aligned}\bar{\Psi}_k F'(x_k)^{-1}F(z_k) &= F'(y_k)^{-1}F(z_k) + 2F'(x_k)^{-1}F(z_k)\Theta_k^2 + O(h^3)\\ &= F'(y_k)^{-1}(\mathbf{1} + 2\Theta_k^2)F(z_k) + O(h^3),\end{aligned}$$

in which we used $F'(x_k)^{-1} = F'(y_k)^{-1} + O(h)$.

Thus, we obtain $p+3$ order three-step iterations

$$\begin{aligned}y_k &= x_k - F'(x_k)^{-1}F(x_k),\\ z_k &= \phi_p(x_k, y_k),\\ x_{k+1} &= z_k - \Big(\mathbf{1} + 2\Big(\frac{F(y_k)}{F(x_k)}\Big)^2\Big)F'(y_k)^{-1}F(z_k).\end{aligned} \tag{34}$$

If we use $\bar{T}_k$ given by (24), (25) in (34) we obtain a family of eighth-order three-step iterations:

$$\begin{aligned}y_k &= x_k - F'(x_k)^{-1}F(x_k),\\ z_k &= y_k - (dI + \alpha t_k + \beta t_k^2)^{-1}(aI + bt_k + ct_k^2)F'(x_k)^{-1}F(y_k),\\ x_{k+1} &= z_k - \Big(\mathbf{1} + 2\Big(\frac{F(y_k)}{F(x_k)}\Big)^2\Big)F'(y_k)^{-1}F(z_k),\end{aligned} \tag{35}$$

where a, b and d are given by (25). Besides of (31), we can consider the family of three-step iterations:

$$\begin{aligned}y_k &= x_k - \bar{a}F'(x_k)^{-1}F(x_k), \quad \bar{a} \in R, \quad \bar{a} \neq 0,\\ z_k &= y_k - \bar{T}_k F'(x_k)^{-1}F(y_k),\\ x_{k+1} &= z_k - \bar{\Psi}_k F'(x_k)^{-1}F(z_k).\end{aligned} \tag{36}$$

For method (36) holds true:

Theorem 10. *The convergence order of the family of iterations* (36) *equal to seven (eight, when $\bar{a} = 1$) if $\bar{T}_k$ is given by* (27) *and*

$$\bar{\Psi}_k = \mathbf{1} + 2\bar{\Theta}_k + 6\bar{\Theta}_k^2 + 3d_k. \tag{37}$$

where Θ_k and d_k are given by (28) *and* (29) *respectively.*

Proof. By Theorem 8 we prove that $F(z_k) = O(h^p)$, $p = 4$ in case $\bar{a} \neq 1$ and $p = 5$ in case $\bar{a} = 1$. The $p+3$ order of convergence condition (33) of (36) is realized as (37). So the convergence order of iterations equals to seven (eight, when $\bar{a} = 1$). □

The combination of (15), (23), (20), (24), (27)–(29) and (32) (or Theorem 5–8 with Theorem 9, 10) gives us a wide set of iterative methods with convergence order ρ (see Table 3). Below we list only the most effective methods of them.

– The fifth-order methods:

$$\begin{aligned}y_k &= x_k - F'(x_k)^{-1}F(x_k),\\ z_k &= y_k - F'(x_k)^{-1}F(y_k),\\ x_{k+1} &= z_k - \Big(\mathbf{1} + 2\frac{F(y_k)}{F(x_k)}\Big)F'(x_k)^{-1}F(z_k);\end{aligned} \tag{38}$$

Table 3

Iterative methods with convergence order ρ

ρ	$\bar{T}_k$	$\bar{\Psi}_k$	ρ	$\bar{T}_k$	$\bar{\Psi}_k$	ρ	$\bar{T}_k$	$\bar{\Psi}_k$	ρ	$\bar{T}_k$	$\bar{\Psi}_k$
5	(20a)	(32b)	6	(20a)	(32c)	7	(16)	(32c)	8	(20c)	(32c)
	(20b)	(32a)		(20b)	(32b)		(20b)	(32c)		(24)	(32c)
	(16)	(32a)		(22)	(32b)		(22)	(32c)		(27) $a = 1$	(32c)
	(22)	(32a)		(27) $a \neq 1$	(32b)		(27) $a \neq 1$	(32c)		(7c)	(8c)
	(27) $a = 0$	(32a)		(16)	(32b)		(20c)	(32b)			
				(7c)	(32a)		(24)	(32b)			
				(24)	(32a)		(27) $a = 1$	(32b)			

$$
\begin{aligned}
y_k &= x_k - F'(x_k)^{-1}F(x_k),\\
z_k &= y_k - \Big(\mathbf{1} + 2\frac{F(y_k)}{F(x_k)}\Big)F'(x_k)^{-1}F(y_k),\\
x_{k+1} &= z_k - F'(x_k)^{-1}F(z_k),
\end{aligned}
\tag{39}
$$

and

$$
\begin{aligned}
y_k &= x_k - F'(x_k)^{-1}F(x_k),\\
z_k &= x_k - \Big(\mathbf{1} + \frac{F(y_k)}{F(x_k)}\Big)F'(x_k)^{-1}F(x_k),\\
x_{k+1} &= z_k - \Big(\mathbf{1} + 2\frac{F(y_k)}{F(x_k)}\Big)F'(x_k)^{-1}F(z_k).
\end{aligned}
\tag{40}
$$

– The sixth order methods:

$$
\begin{aligned}
y_k &= x_k - F'(x_k)^{-1}F(x_k),\\
z_k &= x_k - \Omega_k(\gamma),\\
x_{k+1} &= z_k - \Big(\mathbf{1} + 2\frac{F(y_k)}{F(x_k)}\Big)F'(x_k)^{-1}F(z_k).
\end{aligned}
\tag{41}
$$

Note that the iteration (2.19) in [8] is a particular case of (41) when $\gamma = 0$.

$$
\begin{aligned}
y_k &= x_k - F'(x_k)^{-1}F(x_k),\\
z_k &= y_k - \frac{F(x_k)^2 + 2F(x_k)F(y_k) + cF(y_k)^2}{F(x_k)^2 + (a-2)F(x_k)F(y_k) + dF(y_k)^2}F'(x_k)^{-1}F(y_k), \quad a, c, d \in \mathbb{R},\\
x_{k+1} &= z_k - \Big(\mathbf{1} + 2\frac{F(y_k)}{F(x_k)}\Big)F'(x_k)^{-1}F(z_k).
\end{aligned}
\tag{42}
$$

Most easy case of (42) is obtained when $a = 2$, $c = d = 0$. Another sixth-order iteration obtained from (34) is

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k),\\ z_k &= y_k - F'(x_k)^{-1}F(y_k),\\ x_{k+1} &= z_k - \Big(\mathbf{1} + 2\Big(\frac{F(y_k)}{F(x_k)}\Big)^2\Big)F'(y_k)^{-1}F(z_k), \end{aligned} \tag{43}$$

because of $\bar{\tau}_k = \mathbf{1}$ in (43) and $F(z_k) = O(h^3)$ (see Table 2).

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k),\\ z_k &= y_k - \Big(\mathbf{1} + \Big(\frac{F(y_k)}{F(x_k)}\Big)^2\Big)F'(y_k)^{-1}F(y_k),\\ x_{k+1} &= z_k - F'(x_k)^{-1}F(z_k). \end{aligned} \tag{44}$$

- The seventh order methods:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k),\\ z_k &= x_k - \Omega_k(\gamma),\\ x_{k+1} &= z_k - \Big(\mathbf{1} + 2\Big(\frac{F(y_k)}{F(x_k)}\Big)^2\Big)F'(y_k)^{-1}F(z_k). \end{aligned} \tag{45}$$

Note that when $\gamma = 1/2$, the iteration (45) converted to iteration (2.18) given in [8].

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k),\\ z_k &= y_k - \frac{F(x_k)^2 + 2F(x_k)F(y_k) + cF(y_k)^2}{F(x_k)^2 + (a-2)F(x_k)F(y_k) + dF(y_k)^2}F'(x_k)^{-1}F(y_k), \quad a, c, d \in \mathbb{R},\\ x_{k+1} &= z_k - \Big(\mathbf{1} + 2\Big(\frac{F(y_k)}{F(x_k)}\Big)^2\Big)F'(y_k)^{-1}F(z_k), \end{aligned} \tag{46}$$

and

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k),\\ z_k &= y_k - \Big(\mathbf{1} + \Big(\frac{F(y_k)}{F(x_k)}\Big)^2\Big)F'(y_k)^{-1}F(y_k),\\ x_{k+1} &= z_k - \Big(\mathbf{1} + 2\frac{F(y_k)}{F(x_k)}\Big)F'(x_k)^{-1}F(z_k). \end{aligned} \tag{47}$$

- The eighth order methods:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k),\\ z_k &= y_k - \Big(\mathbf{1} + \Big(\frac{F(y_k)}{F(x_k)}\Big)^2\Big)F'(y_k)^{-1}F(y_k),\\ x_{k+1} &= z_k - \Big(\mathbf{1} + 2\Big(\frac{F(y_k)}{F(x_k)}\Big)^2\Big)F'(y_k)^{-1}F(z_k), \end{aligned} \tag{48}$$

and method (36) with $\bar{a} = 1$.

Note that in (36) only one inverse of $F'(x_k)$ is required, whereas in other three-step iterations (44), (47), (48), (35) with seventh and eighth-order of convergence two inverses of $F'(x_k)$ and $F'(y_k)$ are required.

Remark 1. Using the generating function method [13] one can easily show that the following replacements

$$\mathbf{1} + 2\Theta_k \Rightarrow \frac{1 + a_1\Theta_k + b_1\Theta_k^2}{1 + (a_1 - 2)\Theta_k + c_1\Theta_k^2}, \quad a_1, b_1, c_1 \in R,$$

$$\mathbf{1} + \Theta_k^2 \Rightarrow \frac{1 + a_2\Theta_k + b_2\Theta_k^2}{1 + a_2\Theta_k + (b_2 - 1)\Theta_k^2}, \quad a_2, b_2 \in R,$$

$$\mathbf{1} + 2\Theta_k^2 \Rightarrow \frac{1 + a_3\Theta_k + b_3\Theta_k^2}{1 + a_3\Theta_k + (b_3 - 2)\Theta_k^2}, \quad a_3, b_3 \in R,$$

in the above mentioned methods are also possible and in this case the convergence order maintained. In this way, we obtain multi-parametric families of iterations.

Note that in [3] the three-step iterations (3), (4), (5) were considered. The original idea is to have the weight functions Q_1 and Q_2 chosen in such a way that the method will be of order higher than 4. But this was not successful as the numerical experiments will show. If we use the operations (2) and (3) in [3] then Q_1 and Q_2 satisfy the conditions (20) and (32). So their method indeed has a sixth order convergence. It should be also pointed out that in [10] another definition of division of vectors was introduced and the extensions of some iterations in multidimensional case were considered by means of matrix X such that $Xa = b$ i.e. $a \overset{X}{\rightarrow} b$. But find such matrix X is not easy task.

4. Computational efficiency

In practice, a method is considered computationally efficient if it has higher convergence order and low computational cost. The computational efficiency index of iterative technique is calculated by [15]

$$E = p^{\frac{1}{C}},$$

where p is order of convergence and $C = d + op$ stands for the total computational cost per iteration, d is the number of function evaluations and op is the number of products and quotients per iteration. We discuss the computational efficiency of the proposed methods and made comparisons between these and existing methods of similar nature. We denote by C_i and E_i the total cost and efficiency index for i-th method. The E_i of the presented iterative methods is given in Table 4.

From Table 4, we see that the iterative method (36) has higher efficiency index. Specific property of our proposed methods is that they are much easier to implement as compared to other methods. In fact, each step of our methods requires to solve only one linear system. So passing from x_k to x_{k+1} is realized by solving three linear systems all together. While all other methods of order $p = 5, 6, 7, 8$ require to solve at least seven or eight linear systems [14, 16]. It makes the algorithms computationally more efficient. Thus, our methods are more simple and guarantee high computational efficiency as compared to other same order existing iterative techniques.

5. Results and discussion

The numerical experiments are carried out to confirm the theoretical results obtained in the previous sections. To get this aim, we consider several test problems, some of them are from real-life problems, e.g., Lane-Emden type equation and 2D Bratu problem.

Table 4

Comparison of computational efficiency

№	methods	p	C_i	E_i
1	(38)	5	$C_1 = \frac{1}{3}n^3 + 3n^2 + \frac{17}{3}n$	$5^{1/C_1}$
2	(39)	5	$C_2 = \frac{1}{3}n^3 + 4n^2 + \frac{17}{3}n$	$5^{1/C_2}$
3	(40)	5	$C_3 = \frac{1}{3}n^3 + 4n^2 + \frac{14}{3}n$	$5^{1/C_3}$
4	(41), $\gamma = 0$	6	$C_4 = \frac{1}{3}n^3 + 5n^2 + \frac{16}{3}n$	$6^{1/C_4}$
5	(42), a=2 $c = d = 0$	6	$C_5 = \frac{1}{3}n^3 + 4n^2 + \frac{17}{3}n$	$6^{1/C_5}$
6	(43)	6	$C_6 = \frac{2}{3}n^3 + 5n^2 + \frac{13}{3}n$	$6^{1/C_6}$
7	(44)	6	$C_{11} = \frac{1}{3}n^3 + 4n^2 + \frac{14}{3}n$	$6^{1/C_{11}}$
8	(45)	7	$C_7 = \frac{2}{3}n^3 + 5n^2 + \frac{16}{3}n$	$7^{1/C_7}$
9	(46)	7	$C_8 = \frac{1}{3}n^3 + 5n^2 + \frac{16}{3}n$	$7^{1/C_8}$
10	(47)	7	$C_9 = \frac{1}{3}n^3 + 5n^2 + \frac{16}{3}n$	$7^{1/C_9}$
11	(48)	8	$C_{10} = \frac{2}{3}n^3 + 5n^2 + \frac{13}{3}n$	$8^{1/C_{10}}$
12	(36)	8	$C_{12} = \frac{1}{3}n^3 + 4n^2 + \frac{17}{3}n$	$8^{1/C_{12}}$

The experiments were made with an Intel Core processor i5-4590, with a CPU of 3.30 GHz and 4096 MB of RAM memory. For comparison, we consider the proposed ρ-order methods ($\rho = 5, 6, 7, 8$) and methods proposed in [17], [19], [18] and [14], namely, T_1, PM7, NLM8, and ZMO8, respectively. We also consider the sixth and seventh order methods (2.18) and (2.19) in [8]. In Tables 5–8, we give the error between two consecutive iterations $\|x_k - x_{k-1}\|$, computational order of convergence ρ_c (see [17, 19]) is given by

$$\rho_c = \frac{\ln(\|x_{k+1} - x_k\|/\|x_k - x_{k-1}\|)}{\ln(\|x_k - x_{k-1}\|/\|x_{k-1} - x_{k-2}\|)}.$$

In addition, we include the elapsed CPU time (in seconds) in these tables. For each case, the following stopping criterion is used to terminate the iterations:

$$\|x_k - x_{k-1}\|_2 \leq 10^{-150}.$$

Example 1. As a first example, we have taken the following small system (see [8]):

$$\begin{aligned} x_{(1)}x_{(2)} + x_{(3)}(x_{(2)} + x_{(4)}) &= 2, \\ x_{(1)}x_{(3)} + x_{(2)}(x_{(1)} + x_{(4)}) &= 1, \\ x_{(1)}x_{(4)} + x_{(3)}(x_{(1)} + x_{(2)}) &= 3, \\ x_{(3)}x_{(2)} + x_{(1)}(x_{(2)} + x_{(4)}) &= 1. \end{aligned}$$

The initial vector is $x_0 = \{-4, -3, -6, -6\}^T$ for the solution $x^* = \{-1.04, 0.26, -1.64, -1.64\}^T$.

Table 5

Comparison of methods for Example 1

Methods	k	$\|x_k - x_{k-1}\|$	ρ_c	e-time
(38)	4	0.9003e-321	4.99	0.9486
(39)	4	0.1245e-342	4.99	0.9656
(40)	4	0.1983e-306	4.99	1.2112
SSK5 [22]	4	0.3751e-433	4.99	1.2507
T_1 [17]	4	0.4856e-591	4.99	1.2586
(41), $\gamma = 1/2$	4	0.4756e-801	5.99	1.6553
(41), $\gamma = 0$ [8]	4	0.1132e-714	5.99	1.3688
(42), a=2 $c = d = 0$	4	0.7142e-602	5.99	1.1804
(43)	4	0.7623e-634	5.99	1.3362
(44)	4	0.4561e-711	5.99	1.2558
(45), $\gamma = 0$	4	0.2456e-1002	6.99	1.4639
(45), $\gamma = 1/2$ [8]	4	0.3452e-1220	6.99	1.1154
(46), a=2 $c = d = 0$	4	0.1138e-1009	6.99	1.0325
(47)	4	0.1988e-1179	6.99	1.7041
PM7 [19]	4	0.7145e-1051	6.99	1.9620
(48)	4	0.9001e-2007	7.99	1.5592
(36)	4	0.8124e-4832	7.99	1.1801
NLM8 [18]	4	0.4356e-2089	7.99	1.7416
ZMO8 [14]	4	0.9140e-1991	7.99	1.1874

Example 2. Consider a system with 20 equations (see [6, 19]):

$$x_{(i)} - \cos\left(2x_{(i)} - \sum_{j=1}^{4} x_{(j)}\right) = 0,$$
$$i = 1, 2, \dots, 20.$$

The solution of this system is $x^* = \{0.5149, 0.5149, \dots, 0.5149\}^T$. We choose the initial approximation $x_0 = \{1, 1, \dots, 1\}^T$ for obtaining the solution x^*.

Example 3. We consider the singular boundary value problem (SBVP) [23]:

$$u''(x) + \frac{2}{x}u'(x) + \sin u(x) = 0, \quad u(0) = 1, \quad u'(0) = 0.$$

Table 6

Comparison of methods for Example 2

Methods	k	$\|x_k - x_{k-1}\|$	ρ_c	e-time
(38)	4	0.6894e-589	4.99	15.838
(39)	4	0.2611e-306	4.99	15.837
(40)	4	0.6894e-589	4.99	15.791
SSK5 [22]	4	0.5781e-568	4.99	23.215
T_1 [17]	4	0.4856e-591	4.99	24.622
(41), $\gamma = 1/2$	4	0.1374e-1121	5.99	12.036
(41), $\gamma = 0$ [8]	4	0.7735e-1156	5.99	12.823
(42), a=2 $c = d = 0$	4	0.8521e-1107	5.99	11.022
(43)	4	0.9195e-1225	5.99	15.005
(44)	4	0.9353e-1292	5.99	12.828
(45), $\gamma = 0$	4	0.3459e-2153	6.99	16.432
(45), $\gamma = 1/2$ [8]	4	0.5977e-2102	6.99	18.955
(46), a=2 $c = d = 0$	4	0.9521e-2080	6.99	11.885
(47)	4	0.1142e-2181	6.99	11.891
PM7 [19]	4	0.4326e-2298	6.99	37.862
(48)	4	0.1717e-3730	7.99	8.9289
(36)	4	0.7355e-4562	8.00	8.1871
NLM8 [18]	4	0.7145e-4451	7.99	26.246
ZMO8 [14]	4	0.9784e-4962	7.99	26.384

After applying finite difference formulas the problem is reduced to a system of $n-1$ nonlinear equations with $n-1$ unknowns:

$$\frac{u_{k-1} - 2u_k + u_{k+1}}{h^2} + \frac{1}{x_k}\left(\frac{u_{k-1} - u_{k+1}}{h}\right) + \sin u_k = 0, \quad k = 1, 2, 3, \dots n-1.$$

We set $n = 101$ and take the initial guess $u_0 = (0.2, 0.2, \dots, 0.2)^T$.

Example 4. We consider the 2D Bratu problem [19]:

$$\begin{aligned} &u_{xx} + u_{yy} + \lambda e^u = 0, \\ &\Omega : \{(x, y) \in 0 \le x \le 1,\ 0 \le y \le 1\}, \\ &u = 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where $u = u(x, y)$.

Table 7

Comparison of methods for Example 3

Methods	k	$\|x_k - x_{k-1}\|$	ρ_c	e-time
(38)	4	0.6894e-419	4.99	33.349
(39)	4	0.2611e-400	4.99	33.353
(40)	4	0.6894e-409	4.99	32.975
SSK5 [22]	4	0.7561e-457	4.99	42.121
T_1 [17]	4	0.8245e-492	4.99	41.714
(41), $\gamma = 1/2$	4	0.2456e-1007	5.99	64.782
(41), $\gamma = 0$ [8]	4	0.4781e-1089	5.99	64.209
(42), a=2 $c = d = 0$	4	0.8521e-1107	5.99	29.247
(43)	4	0.2145e-1059	5.99	43.619
(44)	4	0.9353e-1020	5.99	61.839
(45)	4	0.3785e-1992	6.99	62.167
(45), $\gamma = 1/2$ [8]	4	0.5977e-1988	6.99	61.453
(46), a=2 $c = d = 0$	4	0.9521e-1997	6.99	59.098
(47)	4	0.1142e-1999	6.99	59.917
PM7 [19]	4	0.1756e-2001	6.99	101.86
(48)	4	0.1717e-2650	7.99	63.181
(36)	4	0.2751e-5821	8.00	58.567
NLM8 [18]	4	0.1457e-2775	7.99	120.09
ZMO8 [14]	4	0.6789e-2811	7.99	121.55

Applying the finite-difference formulas the problem is reduced to the nonlinear systems:

$$u_{i,j+1} - 4u_{i,j} + u_{i+1,j} + u_{i,j-1} + u_{i-1,j} + h^2\lambda e^{u_{i,j}} = 0,$$

where $u_{i,j}$ is u at (x_i, y_j) and $1 \le i, j \le N$. For obtaining a large nonlinear system of size 100×100, we take $N = 11$ and $\lambda = 0.1$. The initial vector is $u_0 = 0.1\big(\sin(\pi x_1)\sin(\pi y_1),\ \sin(\pi x_2)\sin(\pi y_2), \ldots, \sin(\pi x_{10})\sin(\pi y_{10})\big)^T$ for the nonlinear system.

As can be observed from the Tables 5–8, the performance of the proposed methods is better than that of existing methods in terms of accuracy and CPU time. The comparison for considered problems shows that our method (36) is the fastest as compared to the other methods. The main reason is that for method (36) the inverse of F' is used only once in per iteration.

Table 8

Comparison of methods for Example 4

Methods	k	$\|x_k - x_{k-1}\|$	ρ_c	e-time
(38)	4	0.4215e-655	4.99	50.452
(39)	4	0.7653e-696	4.99	50.955
(40)	4	0.4579e-678	4.99	49.857
SSK5 [22]	4	0.1658e-622	4.99	60.921
T_1 [17]	4	0.4901e-602	4.99	62.103
(41), $\gamma = 1/2$	4	0.2145e-812	5.99	83.325
(41), $\gamma = 0$ [8]	4	0.3457e-805	5.99	111.35
(42), a=2	4	0.9114e-833	5.99	57.112
$c = d = 0$				
(43)	4	0.1342e-799	5.99	82.717
(44)	4	0.2456e-843	5.99	83.840
(45), $\gamma = 0$	4	0.4589e-1411	6.99	82.105
(45), $\gamma = 1/2$ [8]	4	0.9756e-1543	6.99	81.535
(46), a=2	4	0.7946e-1611	6.99	81.397
$c = d = 0$				
(47)	4	0.6789e-1589	6.99	81.912
PM7 [19]	4	0.1456e-1566	6.99	123.82
(48)	4	0.8599e-2316	7.99	85.183
(36)	4	0.6981e-2712	7.99	81.127
NLM8 [18]	4	0.7895e-2178	7.99	143.08
ZCO8 [14]	4	0.4879e-2002	7.99	143.66

Conclusions

The main contributions of this paper are:

- We propose new fourth and fifth order two-step methods for solving the system of nonlinear equations in R^n with the operations of multiplication and division of vectors.
- We extend the well-known two-point optimal fourth-order methods that designed for solving nonlinear equations to the systems of nonlinear equations. These are unexpected and elegant results.
- We also proposed $p\,(5 \le p \le 8)$- order three-step iterative methods for solving the systems of nonlinear equations. These families of methods include some known methods as particular cases. Moreover, if we use generating function methods in these methods we obtain multi-parametric families of iterative methods.

- The proposed methods (exception of (35)) are simple and require solving three linear systems, whereas the existing methods of the same order convergence require to solve at least seven or eight linear systems. Moreover, they based on a multiplicity of vector by vector, instead of matrix-vector multiplication that inherent in other methods. Both these two factors make our algorithms computationally efficient and in principle new approach to construct higher order iterations.
- To illustrate the high efficiency and accuracy of the proposed methods, the numerical experiments are carried out on both academic and real world problems. Finally, based on numerical results, one can conclude that our methods are the most efficient and fastest than the existing ones of similar nature.

Author Contributions: Conceptualization, T. Zhanlav; methodology, T. Zhanlav; software, Kh. Otgondorj; formal analysis, T. Zhanlav, and Kh. Otgondorj; validation, Kh. Otgondorj; writing—original draft preparation, T. Zhanlav, and Kh. Otgondorj; writing—review and editing, Kh. Otgondorj. All authors have read and agreed to the published version of the manuscript.

Funding: The work was supported partially by the Foundation of Science and Technology of Mongolia under Grant Number MAS_2022/09.

Data Availability Statement: No new data were created or analysed during this study. Data sharing is not applicable.

Acknowledgments: The authors wish to thank the editor and the anonymous referees for their valuable suggestions and comments on the first version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Behl, R., Cordero, A., Motsa, S. S. & Torregrosa, J. R. Construction of fourth-order optimal families of iterative methods and their dynamics. *Applied Mathematics and Computation* **271.** doi:10.1016/j.amc.2015.08.113 (2015).
2. Cătinas, E. A survey on the high convergence orders and computational convergence orders of sequences. *Applied Mathematics and Computation* **343.** doi:10.1016/j.amc.2018.08.006 (2019).
3. Changbum, C. & Neta, B. Developing high order methods for the solution of systems of nonlinear equations. *Applied Mathematics and Computation* **344.** doi:10.1016/j.amc.2018.09.032 (2019).
4. Changbum, C. & Neta, B. An efficient derivative-Free method for the solution of systems of equations. *Numerical Functional Analysis and Optimization* **42.** doi:10.1080/01630563.2021.1931313 (2021).
5. Ham, Y. & Changbum, C. A fifth-order iterative method for solving nonlinear equations. *Applied Mathematics and Computation* **194.** doi:10.1016/j.amc.2007.04.005 (2007).
6. Wang, X. Fixed-point iterative method with eighth-order constructed by undetermined parameter technique for solving nonlinear systems. **13.** doi:10.3390/sym13050863 (2021).
7. Cuyt, A. A. & van der Cruyssen, P. Abstract Padé-approximants for the solution of a system of nonlinear equations. *Computers & Mathematics with Applications* **9.** doi:10.1016/0898-1221(83)90119-0 (1983).
8. Dehghan, M. & Shirilord, A. Three-step iterative methods for numerical solution of systems of nonlinear equations. *Engineering with Computers* **38.** doi:10.1007/s00366-020-01072-1 (2020).
9. Su, Q. A unified model for solving a system of nonlinear equations. *Applied Mathematics and Computation* **290.** doi:10.1016/j.amc.2016.05.047 (2016).
10. Ghanbari, B. & Changbum, C. A constructive method for solving the equation $Xa = b$ in R^n: A generalization of division in R^n. *Applied Mathematics and Computation* **364.** doi:10.1016/j.amc.2019.124673 (2020).

11. Zhanlav, T., Chuluunbaatar, O. & Ulziibayar, V. The necessary and sufficient conditions for two and three-point iterative method of Newton's type iterations. *Computational Mathematics and Mathematical Physics* **57.** doi:10.1134/S0965542517070120 (2017).
12. Zhanlav, T., Chun, C., Otgondorj, K. & Ulziibayar, V. High-order iterations for systems of nonlinear equations. *International Journal of Computer Mathematics* **97.** doi:10.1080/00207160.2019.1652739 (2020).
13. Zhanlav, T., Chuluunbaatar, O. & Ulziibayar, V. Generating function method for constructing new iterations. *Applied Mathematics and Computation* **315.** doi:10.1016/j.amc.2017.07.078 (2017).
14. Zhanlav, T., Mijiddorj, R. & Otgondorj, K. A family of Newton-type methods with seventh and eighth-order of convergence for solving systems of nonlinear equations. *Hacettepe Journal of Mathematics and Statistics* **54.** doi:10.15672/hujms.1061471 (2023).
15. Ostrowski, A. M. *Solution of equations and systems of equations* 202 pp. (Academic Press, New York, 1966).
16. Ahmad, F., Soleymani, F., Haghani, F. K. & Serra-Capizzano, S. Higher order derivative-free iterative methods with and without memory for systems of nonlinear equations. *Applied Mathematics and Computation* **314.** doi:10.1016/j.amc.2017.07.012 (2017).
17. Singh, H. & Sharma, J. R. Simple and efficient fifth order solvers for systems of nonlinear problems. *Mathematical Modelling and Analysis* **28.** doi:10.3846/mma.2023.16244 (2023).
18. Sharma, J. & Arora, H. Improved Newton-like methods for solving systems of nonlinear equations. *SeMA Journal* **74.** doi:10.1007/s40324-016-0085-x (2017).
19. Behl, R. & Arora, H. A novel scheme having seventh-order convergence for nonlinear systems. *Journal of Computational and Applied Mathematics* **404.** doi:10.1016/j.cam.2020.113301 (2020).
20. Cordero, A., Rojas-Hiciano, R. V., Torregrosa, J. R. & Vassileva, M. P. A highly efficient class of optimal fourth-order methods for solving nonlinear systems. *Numerical Algorithms* **95.** doi:10.1007/s11075-023-01631-9 (2024).
21. Zhanlav, T., Chun, C. & Otgondorj, K. Construction and dynamics of efficient high-Order methods for nonlinear systems. *International Journal of Computational Methods* **19.** doi:10.1142/S0219876222500207 (2022).
22. Singh, H., Sharma, J. R. & Kumar, S. A simple yet efficient two-step fifth-order weighted-Newton method for nonlinear models. *Numerical Algorithms* **93.** doi:10.1007/s11075-022-01412-w (2023).
23. Parand, K., Dehghan, M., Rezaei, A. R. & Ghaderi, S. M. An approximation algorithm for the solution of the nonlinear Lane–Emden type equations arising in astrophysics using Hermite functions collocation method. *Computer Physics Communications* **181.** doi:10.1016/j.cpc.2010.02.018 (2010).

Information about the authors

Tugal Zhanlav—Academician, Professor, Doctor of Sciences in Physics and Mathematics, (e-mail: tzhanlav@yahoo.com, ORCID: 0000-0003-0743-5587, Scopus Author ID: 24484328800)

Khuder Otgondorj—Associate Professor of Department of Mathematics at School of Applied Sciences, Mongolian University of Science and Technology (e-mail: otgondorj@gmail.com, ORCID: 0000-0003-1635-7971, Scopus Author ID: 57209734799)

УДК 519.872:519.217
PACS 07.05.Tp, 02.60.Pn, 02.70.Bf
DOI: 10.22363/2658-4670-2024-32-4-425-444 EDN: DIOZWP

Разработка и адаптация итерационных методов высшего порядка в R^n с конкретными правилами

Т. Жанлав[1,2], Х. Отгондорж[2]

[1] Институт математики и цифровой технологии, Монгольская академия наук, Улан-Батор, 13330, Монголия

[2] Монгольский государственный университет науки и технологии, Улан-Батор, 14191, Монголия

Аннотация. В данной работе мы предлагаем двухшаговые итерационные методы четвёртого и пятого порядков для решения систем нелинейных уравнений в R^n с использованием операций векторного умножения и деления. Некоторые из предложенных оптимальных методов четвёртого порядка рассматриваются как расширение известных методов, разработанных исключительно для решения нелинейных уравнений. Мы также разработали трёхточечные итерационные методы p-порядка ($5 \leq p \leq 8$) для решения систем нелинейных уравнений, которые включают некоторые известные итерации как частные случаи. Проведён расчёт и сравнение вычислительной эффективности новых методов. Представлены результаты численных экспериментов для подтверждения теоретических выводов относительно порядка сходимости и вычислительной эффективности. Сравнительный анализ демонстрирует превосходство разработанных численных методов.

Ключевые слова: нелинейные системы, методы типа Ньютона, порядок сходимости, вычислительная эффективность, трёхшаговая итерация