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Solving a two-point second-order LODE problem by constructing a complete system of solutions using a modified Chebyshev collocation method

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Abstract. Earlier we developed a stable fast numerical algorithm for solving ordinary differential equations of the first order. The method based on the Chebyshev collocation allows solving both initial value problems and problems with a fixed condition at an arbitrary point of the interval with equal success. The algorithm for solving the boundary value problem practically implements a single-pass analogue of the shooting method traditionally used in such cases. In this paper, we extend the developed algorithm to the class of linear ODEs of the second order. Active use of the method of integrating factors and the d'Alembert method allows us to reduce the method for solving second-order equations to a sequence of solutions of a pair of first-order equations. The general solution of the initial or boundary value problem for an inhomogeneous equation of the second order is represented as a sum of basic solutions with unknown constant coefficients. This approach ensures numerical stability, clarity, and simplicity of the algorithm.

Key words and phrases: linear ordinary differential equation of the second order, stable method, Chebyshev collocation method, d'Alembert method, integrating factor

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1. Introduction

The paper studies a method for solving linear ordinary differential equations (ODEs) of the second order using integrating factors [1–3]. The method of integrating factors in combination with the Chebyshev collocation method [4] was previously applied by the authors to solve first-order ODEs (of general form) [5]. Moreover, the Chebyshev collocation method was successfully applied by the authors to solve second-order linear ODEs (LODEs) using both differentiation matrices [6] and integration matrices [7]. K.P. Lovetskiy et al. developed and applied a modified Chebyshev collocation method, which turned out to be not only more reliable, but also significantly more efficient compared to previous versions of the collocation method and other Runge–Kutta-type methods (see [5–9]) or shooting method [10].

At the first stage of the two-stage modified method proposed by the authors, when expanding the approximate solution in Chebyshev polynomials (of the first or second kind), the corresponding special Gauss–Chebyshev–Lobatto grids are used, on which the search for a part of the coefficients of the approximate general solution of the ODE is reduced to solving non-degenerate and well-conditioned (with diagonal matrices) System of Linear Algebraic Equations (SLAE). At the second stage, the solution is refined by using correctly formulated initial (or boundary) conditions. In this case, the SLAE with a positive definite diagonal matrix is solved first, and then the low-dimensional (one- or two-dimensional) SLAE is solved with respect to the first coefficients of the expansion of the solution in Chebyshev polynomials. The method allows solving with equal efficiency both initial problems and problems with conditions at arbitrary points, previously solved, e.g., by the shooting method, which thus loses its relevance.

Thus, we propose a constructive algorithm for approximate numerical solution of a wide class of LODEs. At the same time, the stage of the algorithm, consisting of solving the SLAE with a diagonal matrix, actually does not require computational costs, because it is reduced to a set of a small number of the simplest computational procedures. And only at the final stage, comprising the calculation of the first pair of coefficients of the expansion of the final particular solution, it is necessary to solve two-dimensional linear algebraic systems of equations determined by the initial or boundary conditions.

The method has proven itself to be perfect in solving one-point problems for first-order ODEs (see [5, 8, 9]). The application of the modified Chebyshev collocation method to solving second-order ODEs has also demonstrated high efficiency. We solve two-point problems for second-order linear ODEs using [11] the two-stage Chebyshev collocation method. The first stage is devoted to finding an approximate solution to the ODE in the form of a Chebyshev polynomial [12] with undetermined first coefficients. At the second stage, the first coefficients (if they exist) are found by solving a 2×2 SLAE [5–7, 13]. The first stage can be implemented in several not entirely equivalent ways [14]. Ref. [6] presents the Chebyshev collocation method for obtaining a solution to a second order LODE using the Chebyshev differentiation matrix [15]. The paper [7] implements the Chebyshev collocation method for obtaining a solution to a second order LODE using the Chebyshev antidifferentiation matrix. The authors noted that constructing a general (complete) solution from the individual partial solutions of the LODE obtained in this way seems to be a computationally complex task. At the same time, using an intermediate method that makes use of integrating factors to reduce the LODE to the form of a total derivative allows one to obtain general (complete) solutions of the second-order LODE more efficiently.

In the present article we seek approximate solutions of linear second-order ODEs of a rather general form

$$a(x)y''_{xx} + b(x)y'_x + c(x)y + f(x) = 0, \tag{1}$$

by the Chebyshev collocation method

2. Methods and algorithms

Let us consider step-by-step the methods for calculating contributions to the complete solution of a second order LODE. In each specific case, the solution of the original problem is divided into two stages. At the first stage, we find out the conditions that the coefficients of the second-order equations under study must satisfy, allowing us to reduce the search for the first of a pair of linearly independent solutions of a second-order linear equation to the solution of a first-order equation.

It turns out that such conditions can be determined at least for

- linear ordinary differential equations with constant coefficients;
- exact linear ordinary differential equations;
- linear equations reducible to a total differential form by means of integrating factors.

In the case of a homogeneous equation with constant non-zero coefficients a, b, c linearly independent solutions of such an ODE can be found directly using the characteristic equation [16, 17]. In the worst case, i.e., when the discriminant $b^2 - 4ac$ is equal to zero, at least one of the solutions is easily determined.

In this case, the corresponding homogeneous equation takes the form

$$ay''(x) + by'(x) + \frac{b^2}{4a}y(x) = 0,$$

from which it follows that the characteristic equation allows finding only one solution

$$y_1(x) = e^{\frac{-b}{2a}x}.$$

If the coefficient functions depend continuously on the argument, a theoretical study of the conditions for reducing the second order LODE to the form of a full derivative of the first-order LODE is given below, in Section 3. The conditions that the coefficients of the inhomogeneous equation must satisfy for the possible construction of the potential are investigated. When sufficient conditions are met, a particular solution of the homogeneous first order LODE is constructed, which becomes the first necessary basic solution of the main nonhomogeneous second order LODE.

After obtaining the first linearly independent solution $y_1(t)$ of the second order LODE, at the next step, using several known algorithms [16–19], one can find the second linearly independent solution $y_2(t)$ and, consequently, the general solution. The most general and convenient method for finding the second solution numerically is the order reduction method [19, 20] (d'Alembert reduction).

Let one solution $y_1(t)$ of the linear homogeneous equation of the second order (1) be known and it is required to find the second linearly independent solution $y_2(t)$, thereby constructing a fundamental system of solutions of the inhomogeneous equation [16, 19]. For the brevity of presenting the method, we introduce the notation $p(x) = \dfrac{b(x)}{a(x)}$, $q(x) = \dfrac{c(x)}{a(x)}$, $g(x) = \dfrac{-f(x)}{a(x)}$.

Equation (1) takes the form

$$y'' + p(x)y' + q(x)y = g(x).$$

When the solution $y_1(t)$ of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0, \tag{2}$$

is found, we find another linear independent solution of the nonhomogeneous equation in the form of a product of the first solution $y_1(x)$ and an unknown function $v(x)$:

$$y(x) = y_1(x)v(x).$$

The search for the solution in the form of a product of the known solution $y_1(x)$ of the homogeneous equation (2) and a non-constant function $v(x)$ is explained by the fact that such a product is guaranteed to be a function linearly independent of $y_1(x)y$ and thus restricts the search for $y(x)$ to a one-dimensional subspace of the space of solutions of our ODE that is not covered by $y_1(x)$.

Actually, such an approach allows finding a general solution to an inhomogeneous equation. Namely, substituting $y_1(x)v(x)$ into $y'' + p(x)y' + q(x)y = g(x)$, and taking into account that $y_1(t)$ is a solution to the homogeneous equation, we obtain a nonhomogeneous equation with respect to the unknown derivative of the desired function $v'(x)$:

$$y_1v'' + \left(2y_1' + p(x)y_1\right)v' = g(x).$$

As a result, for calculating the factor $v(x)$ in the second solution (assuming, that $y_1(x) \neq 0$) we obtain a differential equation of the first order with respect to $v'(x)$

$$v'' + \left(2\frac{y_1'(x)}{y_1(x)} + p(x)\right)v' = \frac{g(x)}{y_1(x)}. \tag{3}$$

Applying the technique of solving nonhomogeneous ordinary differential equations of the first order, based on integrating factors and approved in Refs. [5, 21], we get the desired solution

$$v'(x) = V(x)\left[C_1 + \int_{x_0}^{x} \frac{g(t)}{y_1(t)V(t)}dt\right], \tag{4}$$

where, considering that the solution passes through a certain point x_0, the following notation is introduced:

$$V(x) = \left[\frac{y_1'(x_0)}{y_1(x)}\right]^2 \exp\left[\int_{x_0}^{x} p(t)dt\right].$$

By integrating the ODE (4), we calculate the desired function $v(x)$ and become able to determine the solution of the nonhomogeneous equation by substituting into $y(x) = y_1(x)v(x)$. Hence, the general solution of the nonhomogeneous ODE (1) takes the form

$$y(x) = y_1(x)\left[C_2 + C_1\int_{x_0}^{x} V(t)dt + \int_{x_0}^{x} V(t)dt \int_{x_0}^{x} \frac{g(z)}{y_1(z)V(z)}dz\right]. \tag{5}$$

Finally, we obtained the complete (two-parametric family) solution of the inhomogeneous LODE. If it is necessary to solve a Cauchy problem of a boundary value problem with Eq. (1), we apply the second stage of the modified Chebyshev collocation method to calculate the constants C_1 and C_2.

The technique of finding the first coefficients of expansion of the desired solutions in Chebyshev polynomials is described in enough detail in our papers [6, 7, 13] for all kinds of “boundary” conditions: the Dirichlet, Neuman, and Robin ones.

3. The search for the first solution by reducing a linear ODE to the total derivative form

We consider the nonhomogeneous linear ODE of the second order with coefficients depending on the independent variable:

$$a(x)y''_{xx} + b(x)y'_x + c(x)y + f(x) = 0. \tag{6}$$

This equation is exact, if there exists such function $u(x, y, y')$, that

$$a(x)y''_{xx} + b(x)y'_x + c(x)y + f(x) = \frac{du}{dx}. \tag{7}$$

We want to reduce the search for a solution of the linear second-order ODE to the search for a solution of a linear first-order ODE and, therefore, restrict ourselves to a particular case when

$$u = A(x)y' + B(x)y + F(x). \tag{8}$$

By substituting expression (8) for u into Eq. (7), we obtain equality in the form

$$a(x)y''_{xx} + b(x)y'_x + c(x)y + f(x) = A(x)y'' + \big(A'(x) + B(x)\big)y' + B'(x)y + F'.$$

It will be valid for any smooth $y(x)$ then and only then, when the coefficients of the expressions in the left-hand side and in the right-hand side of the equality coincide:

$$\begin{aligned}
&A(x) = a(x);\\
&A'(x) + B(x) = b(x);\\
&B'(x) = c(x);\\
&F'(x) = f(x).
\end{aligned}$$

This system of four equations allows unambiguous determination of $A(x)$, $B(x)$, (x) from the coefficients of linear ODE (6):

$$\begin{aligned}
&A(x) = a(x);\\
&B(x) = b(x) - a'(x);\\
&F(x) = \int f(x)dx,
\end{aligned}$$

provided that one more condition is fulfilled,

$$c(x) = \big(b(x) - a'(x)\big)',$$

which, therefore, is a necessary and sufficient condition for the possibility to represent the linear ODE (6) in the form (7) with linear potential (8). Hence, the following theorem is valid.

Theorem 3. *The linear ODE* (6) *is exact and has a linear potential*

$$a(x)y'' + b(x)y' + c(x)y + f(x) = \frac{d}{dx}\big(A(x)y' + B(x)y + F(x)\big),$$

when and only when the coefficients of the linear ODE (6) *satisfy the condition*

$$c(x) = \big(b(x) - a'(x)\big)'$$

and the potential has the form

$$u(x, y, y') \equiv a(x)y' + \big(b(x) - a'(x)\big)y + \int f(x)dx = \text{const}. \tag{9}$$

Corollary 1. The linear homogeneous ODE

$$a(x)y''_{xx} + b(x)y'_x + c(x)y = 0 \tag{10}$$

is exact and has linear potential

$$a(x)y'' + b(x)y' + c(x)y = \frac{d}{dx}\big(A(x)y' + B(x)y\big),$$

then and only then, when the coefficients of the linear ODE (6) satisfy the condition

$$c(x) = \big(b(x) - a'(x)\big)'. \tag{11}$$

If for a certain second-order equation of the form (6) the condition (11) is fulfilled, then the search for one of its solutions can be reduced to a search for a solution of the linear first-order ODE with an arbitrary constant const

$$Ay' + By + F = \text{const}.$$

Any solution $y_{part}(x)$ of the linear first-order ODE (8) at any value of the constant const is at the same time a solution to the initial equation (10).

Knowing one solution of the linear homogeneous equation of the second order (10), one can find its other linear independent solution using the d'Alembert method.

We have implemented checking of Eq. (8) fulfilment and searching for integral (9) in Sage as function `lsolve`.

Example 1. Consider a LODE

$$y'' + xy' + y + \cos x = 0.$$

The application of

```
sage: lsolve(diff(y, x, 2)+x*diff(y, x)+y+cos(x))
```

returns

$$x + \sin x + y'.$$

Thus, the integration of the initial second-order equation is analytically reduced to the integration of the first order LODE

$$x + \sin x + y' = \text{const}.$$

Now let us assume that Eq. (6) is not exact. In this case it is possible to try searching for an integrating factor $\mu(x)$ such that the equation

$$\mu(x)a(x)y''_{xx} + \mu(x)b(x)y'_x + \mu(x)c(x)y + \mu(x)f(x) = 0 \tag{12}$$

would be exact.

Theorem 4. *After introducing the factor $\mu(x)$ LODE* (6) *becomes exact and possesses a linear potential*

$$\mu(x) \cdot (a(x)y'' + b(x)y' + c(x)y + f(x)) = \frac{d}{dx}(A(x)y' + B(x)y + F(x))$$

then and only then, when the coefficients in the LODE (10) *satisfy the condition*

$$\mu(x)c(x) = \Big(\mu(x)b(x) - \big(\mu(x)a(x)\big)'\Big)'. \tag{13}$$

In this case the potential has the form

$$u(x, y, y') \equiv \mu(x)a(x)y' + \Big(\mu(x)b(x) - \big(\mu(x)a(x)\big)'\Big)y + \int \mu(x)f(x)dx = \text{const}. \tag{14}$$

Any potential solution at any value of the constant is a solution to Eq. (12).

Having one solution of the linear inhomogeneous second-order equation (12) it is possible to find its other linearly independent solution, using the d'Alembert algorithm.

Equation (13) is a homogeneous linear ODE of the second order with respect to $\mu(x)$. If the initial linear ODE is also homogeneous, then it is possible to formulate a very simple method to find the factor.

Corollary 2. If the linear ODE (12) is homogeneous ($f(x) = 0$) and its coefficients satisfy the relation

$$b'(x)a(x) - a'(x)b(x) - c(x)a(x) = 0 \tag{15}$$

then the linear ODE (12) has an integrating factor

$$\mu(x) = \frac{1}{a(x)}. \tag{16}$$

In this case, the potential for the initial linear second-order ODE is given by the linear ODE of the first order

$$u(x, y, y') \equiv y'(x) + \frac{b(x)}{a(x)} y(x) = \text{const.} \tag{17}$$

Proof. By Theorem 2, in order to reduce a second-order LODE to a first-order LODE it is sufficient to find the factor $\mu(x)$ from Eq. (13). Substitution of expression (16) into it leads to relation (15). To determine the coefficients of potential (14) at $f(x) = 0$ we have:

$$A = \mu a = 1;$$

$$B = \mu b - (\mu a)' = \frac{b}{a};$$

$$F = \mu c - (\mu b)' + (\mu a)'' = \frac{c}{a} - \left(\frac{b}{a}\right)' = 0.$$

Any solution of potential (17) at any value of the constant is a solution to Eq. (12).

Having one solution of the linear homogeneous solution of the second order (12), it is possible to find its another linearly independent solution by using the d'Alembert method.

We implemented the checking of the search for the factor in Sage within the function `lsolve` mentioned above. This function checks the fulfilment of condition (15). In the case of success, it divides the LODE by $a(x)$ and finds the first-order LODE by the methods described in Corollary 1. In the case of failure, the system tries to integrate Eq. (13).

Example 2. Consider a LODE

$$(x^2 + 1)(y' + xy)' = 0.$$

The application of

```
sage: lsolve((x^2+3)*diff(diff(y, x)+x*y, x))
```

returns

$$y' + xy.$$

Thus, the integration of the initial second-order equation is analytically reduced to the integration of the first order LODE

$$y' + xy = \text{const.}$$

Example 3. Consider the LODE

$$y'' + y + \sin x = 0.$$

Our function

```
sage: lsolve(diff(y, x, 2)+y+sin(x))
```

returns a family of factors of this equation:

$$K_1 \sin x + K_2 \cos x.$$

It is possible to take any element of this family: the query

```
sage: lsolve((diff(y, x, 2)+y+sin(x))*sin(x))
```

returns the LODE of the first order

$$\sin x y' + \frac{x}{2} - \cos x - \frac{\sin 2x}{4} = \text{const.}$$

4. Results

Previously, the authors proposed a method for finding a solution to a non-homogeneous linear ordinary differential equation of the second order using a modified Chebyshev collocation method using spectral (Chebyshev) matrices of differentiation and anti-differentiation [6, 7]. In this paper, a method for finding a solution to a second-order LODE is implemented by reducing it to the form of a total derivative either directly or using an integrating factor.

The modified Chebyshev collocation method allows one to obtain a complete system of linearly independent solutions to a linear ordinary differential equation of the second order using the d'Alembert method based on one known existing solution and to construct a general solution to a two-point problem for the corresponding second-order LODE in the case where it exists. In this case, the problem of the existence of a solution to a two-point problem for the corresponding second-order LODE is reduced to the problem of the existence of a solution to a two-dimensional system of linear algebraic equations for the first two coefficients in the expansion of the desired solution to the original problem in Chebyshev polynomials using the collocation method on the Chebyshev–Gauss–Lobatto grid.

5. Discussion

The D'Alembert method allows us to derive Eq. (3), the solution of which using integrating factors gives us the factor $v(x)$ of the general solution $y_1(x)v(x)$ of the inhomogeneous equation. As in the previous case, the numerical solution of Eq. (3) with respect to $v'(x)$ is carried out approximately using the Chebyshev collocation method [5, 7]. Integrating expression (4), we obtain $v(x)$ in the form of an interpolation polynomial. Substituting the obtained expression into the product $y_1(x)v(x)$, we obtain the general solution of the inhomogeneous ODE of the second order in the form (5).

The coefficients C_1, C_2 in the general solution formula are further determined based on the initial or boundary conditions defining the initial or boundary value problem for a second-order differential inhomogeneous equation. In the case of the Cauchy problem, the coefficients are always uniquely determined [16, 17]. In the case where a boundary value problem is considered, the system of resulting equations may have an infinite number of solutions, have no solutions, or have a unique solution. This is determined by the coincidence or difference of the ranks of the proper and extended matrices of the SLAE depending on the "boundary conditions". Thus, if a two-point boundary value problem has a solution, we obtain its approximate solution using the proposed approach—reducing the LODE to the total derivative form.

6. Conclusion

The paper considers an approach to solving linear inhomogeneous second-order ODEs based on the d'Alembert method of order reduction. The method allows, given one solution $y_1(x)$ of the complementary homogeneous equation, calculating both the general solution of the homogeneous equation and the general solution of the inhomogeneous equation. The method for obtaining the first solution of the homogeneous linear differential equation remains a difficult problem within this approach.

We have formulated the conditions for reducing the second-order LODE to the form of a total derivative of the solution using the Chebyshev collocation method. In cases where reduction to the form of a total derivative is not immediately possible, we assume, in the future, the use of a numerical method for solving a second-order equation using the method of integrating factors based on the Chebyshev collocation [7] to obtain the first solution of the accompanying first solution of the homogeneous equation.

The paper proposes an algorithm for obtaining the first basic solution of a complementary homogeneous ODE in cases of an equation with constant coefficients, an exact linear ordinary differential equation, or an equation that can be reduced to the form of a total differential using integrating factors. At the first step of the algorithm, the fulfillment of the conditions of belonging to exact equations or the possibility of finding such an integrating factor with which it is possible to reduce the equation to an exact one is checked. When the conditions set out in the corollaries to Theorems 1 and 2 are met, it is possible to construct a potential for a homogeneous equation—a first-order ODE. The solution to the potential equation can be found numerically [5, 21] using the efficient and stable Chebyshev collocation method. It is this solution to the homogeneous equation that is used further in the D'Alembert algorithm as the first known solution $y_1(x)$ of a second-order ODE.

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Решение двухточечной задачи ЛОДУ второго порядка построением полной системы решений модифицированным методом Чебышевской коллокации

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Аннотация. В предыдущих работах мы разработали устойчивый быстрый численный алгоритм для решения обыкновенных дифференциальных уравнений первого порядка. Метод, основанный на чебышевской коллокации, позволяет одинаково успешно решать как начальные задачи, так и с фиксированным условием в произвольной точке интервала. Алгоритм решения краевой задачи практически реализует однопроходный аналог традиционно применяющегося в таких случаях метода стрельбы (Shooting method). В настоящей работе мы расширяем разработанный алгоритм на класс линейных ОДУ второго порядка. Активное использование метода интегрирующих множителей и метода Даламбера позволяет свести метод решения уравнений второго порядка к последовательности решений пары уравнений первого порядка. Общее решение начальной или краевой задачи для неоднородного уравнения 2-го порядка представляется в виде суммы базисных решений с неизвестными постоянными коэффициентами. Такой подход позволяет обеспечить численную устойчивость, наглядность и простоту алгоритма.

Ключевые слова: линейное обыкновенное дифференциальное уравнение второго порядка, устойчивый метод, метод чебышевской коллокации, метод Даламбера, интегрирующий множитель