Discrete & Continuous Models
& Applied Computational Science
ISSN 2658-7149 (Online), 2658-4670 (Print)

2024, 32 (4) 380–394
http://journals.rudn.ru/miph

Research article

UDC 519.872:519.217
PACS 07.05.Tp, 02.60.Pn, 02.70.Bf
DOI: 10.22363/2658-4670-2024-32-4-380-394
EDN: DDNAPR

Asymptotic diffusion analysis of RQ system M/M/1 with unreliable server

Nataliya M. Voronina[1], Svetlana V. Rozhkova[1, 2]

[1] National Research Tomsk Polytechnic University, 30 Lenina Ave, Tomsk, 634050, Russian Federation

[2] National Research Tomsk State University, 36 Lenina Ave, Tomsk, 634050, Russian Federation

(received: July 18, 2024; revised: August 20, 2024; accepted: September 10, 2024)

Abstract. The paper considers a single-line retrial queueing system with an unreliable server. Queuing systems are called unreliable if their servers may fail from time to time and require restoration (repair), only after which they can resume servicing customers. The input of the system is a simple Poisson flow of customers. The service time and uptime of the server are distributed exponentially. An incoming customer try to get service. The server can be free, busy or under repair. The customer is serviced immediately if the server is free. If it is busy or under repair, the customer goes into orbit. And after a random time it tries to get service again. The study is carried out by the method of asymptotically diffusion analysis under the condition of a large delay of requests in orbit. In this work, the transfer coefficient and diffusion coefficient were found and a diffusion approximation was constructed.

Key words and phrases: retrial queuing system, asymptotic diffusion method, unreliable device

For citation: Voronina, N. M., Rozhkova, S. V. Asymptotic diffusion analysis of RQ system M/M/1 with unreliable server. *Discrete and Continuous Models and Applied Computational Science* **32** (4), 380–394. doi: 10.22363/2658-4670-2024-32-4-380-394. edn: DDNAPR (2024).

1. Introduction

Queuing systems with repeated requests are quite often used in various areas of telecommunications. Modern information processing systems often encounter unstable operating conditions, such as overloads, failures, and resource limitations. Under these conditions, conventional retrial queuing (RQ) systems may not be able to process all incoming requests, resulting in lost information and poor performance [1–4].

Repetitive request systems offer a solution to this problem by providing a mechanism for processing requests that cannot be fulfilled immediately. Instead of discarding such requests, they are resubmitted to the queue after a certain time, increasing the likelihood of successful completion of service. The most complete and detailed description of RQ systems and their detailed comparison with classical queuing systems was reflected in [5–7].

There are different types of unreliability. For example, the works [8–10] consider the unreliability of the server as a breakdown. The authors in [11–14] consider an unreliable server with collisions or conflicts during simultaneous access to the server.

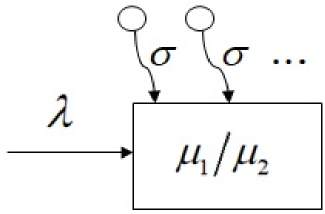

Figure 1. Model of retrial queueing system M/M/1 with unreliable server

This problem is especially relevant when it comes to unreliable servers that can fail due to software errors, hardware malfunctions or external factors. Server failures can lead to data loss, interruption of services, and decreased performance.

If the server fails while servicing the request, it goes to repair. A request under maintenance goes into orbit and awaits recovery of the server. A fairly large number of works are devoted to systems with unreliable server [15–20].

To understand the behavior of systems with repeated requests and evaluate their performance, it is necessary to use analytical methods.

In this paper, we consider a single-line queuing system with an unreliable server. We will conduct the study using the method of asymptotic diffusion analysis. It has been proven that the accuracy of the diffusion approximation exceeds the accuracy of the Gaussian approximation calculated in [21].

2. System description

Any data network, having generated customers, sends them to a shared resource (server). If the server is free, then the customer is served. If the server fails while servicing a customer, it is sent for repair, and the customers go into orbit.

Let's consider an RQ system with an unreliable server, the input of which receives a simple flow of customers with parameter λ. The request is served by the server at a random time, distributed according to an exponential law with the parameter μ_1. An unreliable server can be in one of the following states: idle, busy, or under repair. If the server is idle and an entry customer is received, the server immediately begins servicing the incoming customer. If a customer arrives at a time when the server is busy, then the received customer goes into orbit and waits for the opportunity to occupy the server at the next attempt.

After a random delay, a customer with intensity σ again contacts the device with an attempt to capture it (see Fig. 1). The server's uptime is distributed according to an exponential law with parameter γ_1 if the server is idle, and with parameter γ_2, if the server is busy. As soon as a breakdown occurs, the server is sent for repair. All incoming customers go into orbit. The recovery time after repair is distributed exponentially with the parameter μ_2.

The goal of the work is to study such a system, as well as to find its main characteristics.

3. Kolmogorov equations

Let us denote by $P\{i(t) = i,\ k(t) = k,\ n(t) = n\} = P(k, i, t)$—the probability that at a given time t the server is in state k and in the orbit of i customers. The probability distribution $P(k, i, t)$ satisfies the following system of equations:

$$\begin{aligned}
P_0(i,t+\Delta t) =&(1-\lambda\Delta t)(1-i\sigma\Delta t)(1-\gamma_1\Delta t)P_0(i,t)+\mu_1\Delta tP_1(i,t)+\\
&+\mu_2\Delta tP_2(i,t)+o(\Delta t),\\
P_1(i,t+\Delta t) =&(1-\lambda\Delta t)(1-\mu_1\Delta t)(1-\gamma_2\Delta t)P_1(i,t)+\lambda\Delta tP_0(i,t)+\\
&+\sigma(i+1)\Delta tP_0(i+1,t)+\lambda P_1(i-1,t)+o(\Delta t),\\
P_2(i,t+\Delta t) =&(1-\lambda\Delta t)(1-\mu_2\Delta t)P_2(i,t)+\gamma_1\Delta tP_0(i,t)+\\
&+\gamma_2\Delta tP_1(i-1,t)+\lambda\Delta tP_2(i-1,t)+o(\Delta t).
\end{aligned}$$

Let's create a system of Kolmogorov differential equations:

$$\begin{cases}
\dfrac{\partial P_0(i,t)}{\partial t} = -(\lambda+i\sigma+\gamma_1)P_0(i,t)+\mu_1P_1(i,t)+\mu_2P_2(i,t),\\
\dfrac{\partial P_1(i,t)}{\partial t} = -(\lambda+\mu_1+\gamma_2)P_1(i,t)+\lambda P_0(i,t)+\\
\qquad +\sigma(i+1)P_0(i+1,t)+\lambda P_1(i-1,t),\\
\dfrac{\partial P_2(i,t)}{\partial t} = -(\lambda+\mu_2)P_2(i,t)+\gamma_1P_0(i,t)+\gamma_2P_1(i-1,t)+\lambda P_2(i-1,t).
\end{cases} \tag{1}$$

Let us write down the partial characteristic functions:

$$H_k(u,t)=\sum_{i=0}^{\infty}e^{iuj}P_k(i,t), \quad k=\{0,1,2\},$$

where $j=\sqrt{-1}$.

Multiplying the equations of the system (1) by e^{iuj}, we obtain

$$\begin{cases}
\dfrac{\partial H_0(i,t)}{\partial t} = -(\lambda+\gamma_1)H_0(u,t)+j\sigma e^{ju}\dfrac{\partial H_0(u,t)}{\partial u}+\\
\qquad +\mu_1H_1(u,t)+\mu_2H_2(u,t),\\
\dfrac{\partial H_1(i,t)}{\partial t} = -(\lambda+\mu_1+\gamma_2)H_1(u,t)+\lambda H_0(u,t)-\\
\qquad -j\sigma\dfrac{\partial H_0(u,t)}{\partial u}+\lambda e^{ju}H_1(u,t),\\
\dfrac{\partial H_2(i,t)}{\partial t} = -(\lambda+\mu_2)H_2(u,t)+\gamma_1H_0(u,t)+\\
\qquad +\gamma_2e^{ju}H_1(u,t)+\lambda e^{ju}H_2(u,t).
\end{cases} \tag{2}$$

Summing up the equations of the system (2), we write the equation for the characteristic function

$$H(u,t)=H_0(u,t)+H_1(u,t)+H_2(u,t),$$

then we get

$$\frac{\partial H(u,t)}{\partial t}=(e^{ju}-1)\left(H_1(u,t)(\lambda+\gamma_2)+H_2(u,t)\lambda+j\sigma\frac{\partial H_0(u,t)}{\partial u}\right). \tag{3}$$

We will find a characteristic function of the number of customers in orbit under the condition of a long delay. We will investigate in two stages.

4. Stage 1. Getting the transfer coefficient

Let us introduce the substitutions in the system (2) and the equation (3)

$$\sigma = \varepsilon, \quad \tau = \varepsilon t, \quad u = \varepsilon\omega, \quad H_k(u,t) = F_k(\omega,\tau,\varepsilon), \quad k = \{0,1,2\}.$$

Then we get the following system:

$$\begin{cases} \varepsilon\dfrac{\partial F_0(\omega,\tau,\varepsilon)}{\partial\tau} = -(\lambda+\gamma_1)F_0(\omega,\tau,\varepsilon) + je^{j\varepsilon\omega}\dfrac{\partial F_0(\omega,\tau,\varepsilon)}{\partial\omega} + \\ \qquad + \mu_1 F_1(\omega,\tau,\varepsilon) + \mu_2 F_2(\omega,\tau,\varepsilon), \\ \varepsilon\dfrac{\partial F_1(\omega,\tau,\varepsilon)}{\partial\tau} = -(\lambda+\mu_1+\gamma_2)F_1(\omega,\tau,\varepsilon) + \lambda F_0(\omega,\tau,\varepsilon) - \\ \qquad - j\dfrac{\partial F_0(\omega,\tau,\varepsilon)}{\partial\omega} + \lambda e^{j\varepsilon\omega}F_1(\omega,\tau,\varepsilon), \\ \varepsilon\dfrac{\partial F_2(\omega,\tau,\varepsilon)}{\partial\tau} = -(\lambda+\mu_2)F_2(\omega,\tau,\varepsilon) + \gamma_1 F_0(\omega,\tau,\varepsilon) + \\ \qquad + \gamma_2 e^{j\varepsilon\omega}F_1(\omega,\tau,\varepsilon) + \lambda e^{j\varepsilon\omega}F_2(\omega,\tau,\varepsilon). \end{cases} \tag{4}$$

The equation (3) will take the form:

$$\varepsilon\frac{\partial F(\omega,\tau,\varepsilon)}{\partial\tau} = (e^{j\varepsilon\omega}-1)\left(F_1(\omega,\tau,\varepsilon)(\lambda+\gamma_2) + F_2(\omega,\tau,\varepsilon)\lambda + j\frac{\partial F_0(\omega,\tau,\varepsilon)}{\partial\omega}\right). \tag{5}$$

In the system (4) and the equation (5), we decompose the exponent into a Taylor series:

$$e^{j\omega\varepsilon} = 1 + j\omega\varepsilon, \quad \lim_{\varepsilon\to 0}\frac{e^{j\omega\varepsilon}}{\varepsilon} = j\omega.$$

Let us perform the transition to the limit at $\varepsilon \to 0$, then we obtain:

$$\begin{cases} -(\lambda+\gamma_1)F_0(\omega,\tau) + j\dfrac{\partial F_0(\omega,\tau)}{\partial\omega} + \mu_1 F_1(\omega,\tau) + \mu_2 F_2(\omega,\tau) = 0, \\ -(\mu_1+\gamma_2)F_1(\omega,\tau) + \lambda F_0(\omega,\tau) - j\dfrac{\partial F_0(\omega,\tau)}{\partial\omega} = 0, \\ -\mu_2 F_2(\omega,\tau) + \gamma_1 F_0(\omega,\tau) + \gamma_2 F_1(\omega,\tau) = 0. \end{cases} \tag{6}$$

$$\frac{\partial F(\omega,\tau)}{\partial\tau} = j\omega\left(F_1(\omega,\tau)(\lambda+\gamma_2) + F_2(\omega,\tau)\lambda + j\frac{\partial F_0(\omega,\tau)}{\partial\omega}\right). \tag{7}$$

We will find a solution to the system (6) and the equation (7) in the form:

$$F_k(\omega,\tau) = R_k e^{j\omega x(\tau)}, \quad k = \{0,1,2\},$$

where R_k has the meaning of the stationary probability that the server is in state k, and $x(\tau)$ is a scalar function of the argument τ, which determines at $\varepsilon \to 0$ the average value $\sigma i(\tau/\sigma)$ of the number of customers in orbit normalized by the value at $\varepsilon = \sigma$.

Then the system (6) and the equation (7) will take the form:

$$\begin{cases} -\big(\lambda+\gamma_1+x(\tau)\big)R_0 + \mu_1 R_1 + \mu_2 R_2 = 0, \\ \big(\lambda+x(\tau)\big)R_0 - (\mu_1+\gamma_2)R_1 = 0, \\ \gamma_1 R_0 + \gamma_2 R_1 - \mu_2 R_2 = 0. \end{cases} \tag{8}$$

The probabilities R_k can be found from the system (6) taking into account the normalization condition $R_0 + R_1 + R_2 = 1$.

Since the coefficient of the system of equations (6) depends on x, then R_k can also be written as $R_k(x)$.

$$R_0 = \frac{\mu_2(\mu_1 + \gamma_2)}{(\gamma_1 + \lambda + \mu_2 + x(\tau))\gamma_2 + (\lambda + \mu_1 + x(\tau))\mu_2 + \gamma_1\mu_1},$$
$$R_1 = \frac{(\lambda + x(\tau))\mu_2}{(\gamma_1 + \lambda + \mu_2 + x(\tau))\gamma_2 + (\lambda + \mu_1 + x(\tau))\mu_2 + \gamma_1\mu_1},$$
$$R_2 = \frac{(\lambda + \gamma_1 + x(\tau))\gamma_2 + \gamma_1\mu_1}{(\gamma_1 + \lambda + \mu_2 + x(\tau))\gamma_2 + (\lambda + \mu_1 + x(\tau))\mu_2 + \gamma_1\mu_1}.$$

From the equation (7) we get:

$$x'(\tau) = -x(\tau)R_0 + (\lambda + \gamma_2)R_1 + \lambda R_2.$$

Let us denote the function $a(x) = x'(\tau)$, then

$$a(x) = -x(\tau)R_0 + (\lambda + \gamma_2)R_1 + \lambda R_2,$$

where $a(x)$ is the transfer coefficient.

5. Stage 2. Centering and obtaining the diffusion coefficient

Let us introduce the substitutions in the system (2) and the equation (3)

$$H_k(u,t) = H_k^{(2)}(u,t)e^{j\frac{u}{\sigma}x(t)}, \quad k = \{0,1,2\},$$

we get

$$\begin{cases} \frac{\partial H_0^{(2)}(u,t)}{\partial t} + H_0^{(2)}(u,t)jux'(\sigma t) = -(\lambda + \gamma_1)H_0^{(2)}(u,t) + \\ \quad + j\sigma e^{ju}\left(\frac{\partial H_0^{(2)}(u,t)}{\partial u} + H_0^{(2)}(u,t)j\frac{1}{\sigma}x(\sigma t)\right) + \\ \quad + \mu_1 H_1^{(2)}(u,t) + \mu_2 H_2^{(2)}(u,t) = 0, \\ \frac{\partial H_1^{(2)}(u,t)}{\partial t} + H_1^{(2)}(u,t)jux'(\sigma t) = -(\lambda + \mu_1 + \gamma_2)H_1^{(2)}(u,t) + \\ \quad + \lambda H_0^{(2)}(u,t) - j\sigma\left(\frac{\partial H_0^{(2)}(u,t)}{\partial u} + H_0^{(2)}(u,t)j\frac{1}{\sigma}x(\sigma t)\right) + \\ \quad + \lambda e^{ju}H_1^{(2)}(u,t) = 0, \\ \frac{\partial H_2^{(2)}(u,t)}{\partial t} + H_2^{(2)}(u,t)jux'(\sigma t) = -(\lambda + \mu_2)H_2^{(2)}(u,t) + \\ \quad + \gamma_1 H_0^{(2)}(u,t) + \gamma_2 e^{ju}H_1^{(2)}(u,t) + \lambda e^{ju}H_2^{(2)}(u,t) = 0. \end{cases} \tag{9}$$

$$\frac{\partial H^{(2)}(u,t)}{\partial t} + H^{(2)}(u,t)jux'(\sigma t) = (e^{ju} - 1)\Big(H_1^{(2)}(u,t)(\lambda + \gamma_2) + \\ + H_2^{(2)}(u,t)\lambda + j\sigma\left(\frac{\partial H_0^{(2)}(u,t)}{\partial u} + H_0^{(2)}(u,t)j\frac{1}{\sigma}x(\sigma t)\right)\Big). \tag{10}$$

Let us introduce the substitutions in the system (9) and the equation (10)

$$\sigma = \varepsilon^2, \quad \tau = \varepsilon^2 t, \quad u = \varepsilon\omega, \quad H_k^{(2)}(u,t) = F_k^{(2)}(\omega,\tau,\varepsilon), \quad k = \{0,1,2\}.$$

Then we obtain:

$$\begin{cases} \varepsilon^2 \dfrac{\partial F_0^{(2)}(\omega,\tau,\varepsilon)}{\partial\tau} + F_0^{(2)}(\omega,\tau,\varepsilon) j\varepsilon\omega a(x,\tau) = -(\lambda+\gamma_1) F_0^{(2)}(\omega,\tau,\varepsilon) + \\ \qquad + j\varepsilon e^{j\varepsilon\omega} \dfrac{\partial F_0^{(2)}(\omega,\tau,\varepsilon)}{\partial\omega} - e^{j\varepsilon\omega} F_0^{(2)}(\omega,\tau,\varepsilon) x(\tau) + \\ \qquad + \mu_1 F_1^{(2)}(\omega,\tau,\varepsilon) + \mu_2 F_2^{(2)}(\omega,\tau,\varepsilon), \\ \varepsilon^2 \dfrac{\partial F_1^{(2)}(\omega,\tau,\varepsilon)}{\partial\tau} + F_1^{(2)}(\omega,\tau,\varepsilon) j\varepsilon\omega a(x,\tau) = -(\lambda+\mu_1+\gamma_2)\times \\ \qquad \times F_1^{(2)}(\omega,\tau,\varepsilon) + \lambda F_0^{(2)}(\omega,\tau,\varepsilon) - j\varepsilon \dfrac{\partial F_0^{(2)}(\omega,\tau,\varepsilon)}{\partial\omega} + \\ \qquad + F_0^{(2)}(\omega,\tau,\varepsilon) x(\tau) + \lambda e^{j\varepsilon\omega} F_1^{(2)}(\omega,\tau,\varepsilon), \\ \varepsilon^2 \dfrac{\partial F_2^{(2)}(\omega,\tau,\varepsilon)}{\partial\tau} + F_2^{(2)}(\omega,\tau,\varepsilon) j\varepsilon\omega a(x,\tau) = -(\lambda+\mu_2) F_2^{(2)}(\omega,\tau,\varepsilon) + \\ \qquad + \gamma_1 F_0^{(2)}(\omega,\tau,\varepsilon) + \gamma_2 e^{j\varepsilon\omega} F_1^{(2)}(\omega,\tau,\varepsilon) + \lambda e^{j\varepsilon\omega} F_2^{(2)}(\omega,\tau,\varepsilon). \end{cases} \tag{11}$$

$$\begin{aligned} \varepsilon^2 \frac{\partial F^{(2)}(\omega,\tau,\varepsilon)}{\partial\tau} + F^{(2)}(\omega,\tau,\varepsilon) j\varepsilon\omega a(x,\tau) = (e^{j\varepsilon\omega}-1)\times \\ \times\Big(F_1^{(2)}(\omega,\tau,\varepsilon)(\lambda+\gamma_2) + F_2^{(2)}(\omega,\tau,\varepsilon)\lambda + \\ + j\varepsilon \frac{\partial F_0^{(2)}(\omega,\tau,\varepsilon)}{\partial\omega} - F_0^{(2)}(\omega,\tau,\varepsilon) x(\tau)\Big). \end{aligned} \tag{12}$$

In the system (11), we expand the exponential in a Taylor series and group the terms of order of smallness not higher than ε^2.

$$\begin{cases} F_0^{(2)}(\omega,\tau,\varepsilon) j\varepsilon\omega a(x,\tau) = -(\lambda+\gamma_1) F_0^{(2)}(\omega,\tau,\varepsilon) + \\ \qquad + j\varepsilon(1+j\omega\varepsilon) \dfrac{\partial F_0^{(2)}(\omega,\tau,\varepsilon)}{\partial\omega} - (1+j\omega\varepsilon) F_0^{(2)}(\omega,\tau,\varepsilon) x(\tau) + \\ \qquad + \mu_1 F_1^{(2)}(\omega,\tau,\varepsilon) + \mu_2 F_2^{(2)}(\omega,\tau,\varepsilon) + O(\varepsilon^2), \\ F_1^{(2)}(\omega,\tau,\varepsilon) j\varepsilon\omega a(x,\tau) = -(\lambda+\mu_1+\gamma_2) F_1^{(2)}(\omega,\tau,\varepsilon) + \\ \qquad + \lambda F_0^{(2)}(\omega,\tau,\varepsilon) - j\varepsilon \dfrac{\partial F_0^{(2)}(\omega,\tau,\varepsilon)}{\partial\omega} + F_0^{(2)}(\omega,\tau,\varepsilon) x(\tau) + \\ \qquad + \lambda(1+j\omega\varepsilon) F_1^{(2)}(\omega,\tau,\varepsilon) + O(\varepsilon^2), \\ F_2^{(2)}(\omega,\tau,\varepsilon) j\varepsilon\omega a(x,\tau) = -(\lambda+\mu_2) F_2^{(2)}(\omega,\tau,\varepsilon) + \gamma_1 F_0^{(2)}(\omega,\tau,\varepsilon) + \\ \qquad + \gamma_2(1+j\omega\varepsilon) F_1^{(2)}(\omega,\tau,\varepsilon) + \lambda(1+j\omega\varepsilon) F_2^{(2)}(\omega,\tau,\varepsilon) + O(\varepsilon^2). \end{cases} \tag{13}$$

We will find a solution to the system (13) in the form:

$$F_k^{(2)}(\omega,\tau,\varepsilon) = \Phi(\omega,\tau)(R_k + j\omega\varepsilon f_k) + O(\varepsilon^2), \quad k = \{0,1,2\}, \tag{14}$$

where $R_k = R_k(x,\tau)$.

Substituting expansion (14) into the system (13), we obtain

$$\begin{cases}\Phi(\omega,\tau)j\varepsilon\omega R_0a(x,\tau) = -(\lambda+\gamma_1)\Phi(\omega,\tau)R_0 - (\lambda+\gamma_1)\Phi(\omega,\tau)j\omega\varepsilon f_0 + \\ \qquad + j\varepsilon\dfrac{\partial\Phi(\omega,\tau)}{\partial\omega}R_0 - \Phi(\omega,\tau)R_0x(\tau) - j\omega\varepsilon\Phi(\omega,\tau)R_0x(\tau) - \\ \qquad - \Phi(\omega,\tau)j\omega\varepsilon f_0x(\tau) + \mu_1\Phi(\omega,\tau)R_1 + \mu_1\Phi(\omega,\tau)j\omega\varepsilon f_1 + \\ \qquad + \mu_2\Phi(\omega,\tau)R_2 + \mu_2\Phi(\omega,\tau)j\omega\varepsilon f_2 + O(\varepsilon^2), \\ \Phi(\omega,\tau)j\varepsilon\omega R_1a(x,\tau) = -(\mu_1+\gamma_2)\Phi(\omega,\tau)R_1 + \lambda j\omega\varepsilon\Phi(\omega,\tau)R_1 - \\ \qquad - (\mu_1+\gamma_2)\Phi(\omega,\tau)j\omega\varepsilon f_1 + \lambda\Phi(\omega,\tau)R_0 + \lambda\Phi(\omega,\tau)j\omega\varepsilon f_0 - \\ \qquad - j\varepsilon\dfrac{\partial\Phi(\omega,\tau)}{\partial\omega}R_0 + \Phi(\omega,\tau)R_0x(\tau) + \Phi(\omega,\tau)j\omega\varepsilon f_0x(\tau) + O(\varepsilon^2), \\ \Phi(\omega,\tau)j\varepsilon\omega R_2a(x,\tau) = -\mu_2\Phi(\omega,\tau)R_2 + \lambda j\omega\varepsilon\Phi(\omega,\tau)R_2 - \\ \qquad - \mu_2\Phi(\omega,\tau)j\omega\varepsilon f_2 + \gamma_1\Phi(\omega,\tau)R_0 + \gamma_1\Phi(\omega,\tau)j\omega\varepsilon f_0 + \\ \qquad + \gamma_2\Phi(\omega,\tau)R_1 + \gamma_2 j\omega\varepsilon\Phi(\omega,\tau)R_1 + \gamma_2 j\omega\varepsilon f_1 + O(\varepsilon^2).\end{cases}$$

Taking into account the system (8), we get

$$\begin{cases}\Phi(\omega,\tau)j\varepsilon\omega R_0a(x,\tau) = -(\lambda+\gamma_1)\Phi(\omega,\tau)j\omega\varepsilon f_0 + j\varepsilon\dfrac{\partial\Phi(\omega,\tau)}{\partial\omega}R_0 - \\ \qquad - j\omega\varepsilon\Phi(\omega,\tau)R_0x(\tau) - \Phi(\omega,\tau)j\omega\varepsilon f_0x(\tau) + \mu_1\Phi(\omega,\tau)j\omega\varepsilon f_1 + \\ \qquad + \mu_2\Phi(\omega,\tau)j\omega\varepsilon f_2 + O(\varepsilon^2), \\ \Phi(\omega,\tau)j\varepsilon\omega R_1a(x,\tau) = \lambda j\omega\varepsilon\Phi(\omega,\tau)R_1 - (\mu_1+\gamma_2)\Phi(\omega,\tau)j\omega\varepsilon f_1 + \\ \qquad + \lambda\Phi(\omega,\tau)j\omega\varepsilon f_0 - j\varepsilon\dfrac{\partial\Phi(\omega,\tau)}{\partial\omega}R_0 + \Phi(\omega,\tau)j\omega\varepsilon f_0x(\tau) + O(\varepsilon^2), \\ \Phi(\omega,\tau)j\varepsilon\omega R_2a(x,\tau) = \lambda j\omega\varepsilon\Phi(\omega,\tau)R_2 - \mu_2\Phi(\omega,\tau)j\omega\varepsilon f_2 + \\ \qquad + \gamma_1\Phi(\omega,\tau)j\omega\varepsilon f_0 + \gamma_2 j\omega\varepsilon\Phi(\omega,\tau)R_1 + \\ \qquad + \gamma_2 j\omega\varepsilon f_1 + O(\varepsilon^2).\end{cases} \tag{15}$$

Dividing the equations of the system (15) by $j\omega\varepsilon\Phi(\omega,\tau)$ at $\varepsilon \to 0$, we obtain

$$\begin{cases}R_0a(x,\tau) = -\big(\lambda+\gamma_1+x(\tau)\big)f_0 + \dfrac{\partial\Phi(\omega,\tau)/\partial\omega}{\omega\Phi(\omega,\tau)}R_0 - R_0x(\tau) + \\ \qquad + \mu_1f_1 + \mu_2f_2, \\ R_1a(x,\tau) = \lambda R_1 - (\mu_1+\gamma_2)f_1 + \big(\lambda+x(\tau)\big)f_0 - \dfrac{\partial\Phi(\omega,\tau)/\partial\omega}{\omega\Phi(\omega,\tau)}R_0, \\ R_2a(x,\tau) = \lambda R_2 - \mu_2f_2 + \gamma_1f_0 + \gamma_2R_1 + \gamma_2f_1.\end{cases} \tag{16}$$

The inhomogeneous system (16) corresponds to the homogeneous system (8), therefore we will seek a solution to the system (16) in the form:

$$f_k = CR_k + g_k - \varphi_k\frac{\partial\Phi(\omega,\tau)/\partial\omega}{\omega\Phi(\omega,\tau)}, \quad k = \{0,1,2\}. \tag{17}$$

Substituting the equation (17) into the system (16), we obtain systems with respect to φ_k and g_k:

$$\begin{cases}-\big(\lambda+\gamma_1+x(\tau)\big)\varphi_0 + \mu_1\varphi_1 + \mu_2\varphi_2 = R_0, \\ \big(\lambda+x(\tau)\big)\varphi_0 - (\mu_1+\gamma_2)\varphi_1 = -R_0, \\ \gamma_1\varphi_0 + \gamma_2\varphi_1 - \mu_2\varphi_2 = 0.\end{cases} \tag{18}$$

$$\begin{cases} -\big(\lambda+\gamma_1+x(\tau)\big)g_0+\mu_1 g_1+\mu_2 g_2=\big(a(x,\tau)+x(\tau)\big)R_0,\\ \big(\lambda+x(\tau)\big)g_0-(\mu_1+\gamma_2)g_1=\big(a(x,\tau)-\lambda\big)R_1,\\ \gamma_1 g_0+\gamma_2 g_1-\mu_2 g_2=\big(a(x,\tau)-\lambda\big)R_2-\gamma_2 R_1. \end{cases} \tag{19}$$

If we differentiate the equations of the system (8) by x, then the resulting equations are identical to the equations of the system (16), from which we can conclude that in the system (18) the following equalities are satisfied:

$$\varphi_k=\varphi_k(x,\tau)=\frac{\partial R_k(x,\tau)}{\partial x},\quad \varphi_0+\varphi_1+\varphi_2=0.$$

Let us consider the system (19), which has an infinite number of solutions, since the determinant of the system matrix is equal to zero, and the rank of the system matrix coincides with the rank of the extended matrix of the system.

To find a solution to the system, we add an additional condition $g_0+g_1+g_2=0$ to the system (19) and obtain:

$$g_0=\frac{(-\mu_1-\gamma_2)\big(a(x)+x(\tau)\big)R_0+R_1(\mu_1-\mu_2)\big(\lambda-a(x)\big)}{\big(\lambda+x(\tau)+\gamma_1+\mu_2\big)\gamma_2+(\gamma_1+\mu_2)\mu_1+\mu_2\big(\lambda+x(\tau)\big)},$$

$$g_1=\frac{\big(\lambda+x(\tau)+\gamma_1+\mu_2\big)\big(\lambda-a(x)\big)R_1-R_0\big(\lambda+x(\tau)\big)\big(a(x)+x(\tau)\big)}{\big(\lambda+x(\tau)+\gamma_1+\mu_2\big)\gamma_2+(\gamma_1+\mu_2)\mu_1+\mu_2\big(\lambda+x(\tau)\big)},$$

$$g_2=\frac{\big(\gamma_2+\lambda+\mu_1+x(\tau)\big)\big(a(x)+x(\tau)\big)R_0+R_1\big(\lambda+x(\tau)+\gamma_1+\mu_1\big)\big(a(x)-\lambda\big)}{\big(\lambda+x(\tau)+\gamma_1+\mu_2\big)\gamma_2+(\gamma_1+\mu_2)\mu_1+\mu_2\big(\lambda+x(\tau)\big)}.$$

Let's return to the equation (12). In this equation we group terms of order of smallness not higher than ε^2.

$$\varepsilon^2\frac{\partial F^{(2)}(\omega,\tau,\varepsilon)}{\partial\tau}+F^{(2)}(\omega,\tau,\varepsilon)j\varepsilon\omega a(x,\tau)=\left(j\omega\varepsilon+\frac{(j\omega\varepsilon)^2}{2}\right)\times$$
$$\times\left(F_1^{(2)}(\omega,\tau,\varepsilon)(\lambda+\gamma_2)+F_2^{(2)}(\omega,\tau,\varepsilon)\lambda+j\varepsilon\frac{\partial F_0^{(2)}(\omega,\tau,\varepsilon)}{\partial\omega}-\right.$$
$$\left.-F_0^{(2)}(\omega,\tau,\varepsilon)x(\tau)\right).$$

$$\begin{aligned}\varepsilon^2\frac{\partial F^{(2)}(\omega,\tau,\varepsilon)}{\partial\tau}+F^{(2)}(\omega,\tau,\varepsilon)j\varepsilon\omega a(x,\tau)&=j\omega\varepsilon F_1^{(2)}(\omega,\tau,\varepsilon)(\lambda+\gamma_2)+\\ &+j\omega\varepsilon F_2^{(2)}(\omega,\tau,\varepsilon)\lambda+j^2\omega\varepsilon^2\frac{\partial F_0^{(2)}(\omega,\tau,\varepsilon)}{\partial\omega}-j\omega\varepsilon F_0^{(2)}(\omega,\tau,\varepsilon)x(\tau)+\\ &+\frac{(j\omega\varepsilon)^2}{2}F_1^{(2)}(\omega,\tau,\varepsilon)(\lambda+\gamma_2)+\frac{(j\omega\varepsilon)^2}{2}F_2^{(2)}(\omega,\tau,\varepsilon)\lambda-\\ &-\frac{(j\omega\varepsilon)^2}{2}F_0^{(2)}(\omega,\tau,\varepsilon)x(\tau)+O(\varepsilon^3).\end{aligned} \tag{20}$$

Substituting expansion (14) into the equation (20), we obtain

$$\begin{aligned}&\varepsilon^2\frac{\partial\Phi(\omega,\tau,\varepsilon)}{\partial\tau}+j\varepsilon\omega a(x,\tau)\Phi(\omega,\tau)\big(1+j\omega\varepsilon(f_0+f_1+f_2)\big)=\\ &=j\omega\varepsilon\left((\lambda+\gamma_2)R_1+\lambda R_2-x(\tau)R_0+j\varepsilon\frac{\partial\Phi(\omega,\tau,\varepsilon)}{\partial\omega}R_0\right)\Phi(\omega,\tau)+\\ &+\frac{(j\omega\varepsilon)^2}{2}((\lambda+\gamma_2)R_1+\lambda R_2-x(\tau)R_0)\Phi(\omega,\tau)+\\ &+(j\omega\varepsilon)^2((\lambda+\gamma_2)f_1+\lambda f_2-x(\tau)f_0)\Phi(\omega,\tau)+O(\varepsilon^3).\end{aligned} \tag{21}$$

Taking into account that $a(x) = -x(\tau)R_0 + (\lambda + \gamma_2)R_1 + \lambda R_2$, we eliminate the terms of the order of smallness ε in the equation (21). Then we reduce by ε^2 and perform the transition to the limit at $\varepsilon \to 0$.

$$\begin{aligned}
&\frac{\partial \Phi(\omega,\tau)}{\partial \tau} + (j\omega)^2 a(x,\tau)\Phi(\omega,\tau)(f_0 + f_1 + f_2) = \\
&= (j\omega)^2\Big(\frac{\partial \Phi(\omega,\tau)/\partial\omega}{\omega} R_0 + (\lambda+\gamma_2) f_1 + \lambda f_2 - x(\tau) f_0\Big)\Phi(\omega,\tau) + \\
&\quad + \frac{(j\omega\varepsilon)^2}{2} a(x,\tau)\Phi(\omega,\tau).
\end{aligned} \tag{22}$$

Substituting the equation (17) into the equation (22), we obtain the following equation:

$$\begin{aligned}
&\frac{\partial \Phi(\omega,\tau)}{\partial \tau} + (j\omega)^2 a(x,\tau)\Phi(\omega,\tau) = (j\omega)^2\Phi(\omega,\tau)\Big((\lambda+\gamma_2)\Big(CR_1 + g_1 - \\
&- \varphi_1 \frac{\partial \Phi(\omega,\tau)/\partial\omega}{\omega\Phi(\omega,\tau)}\Big) + \lambda\Big(CR_2 + g_2 - \varphi_2\frac{\partial \Phi(\omega,\tau)/\partial\omega}{\omega\Phi(\omega,\tau)}\Big) - x(\tau)\Big(CR_0 + \\
&+ g_0 - \varphi_0\frac{\partial \Phi(\omega,\tau)/\partial\omega}{\omega\Phi(\omega,\tau)}\Big) + \frac{\partial \Phi(\omega,\tau)/\partial\omega}{\omega\Phi(\omega,\tau)} R_0\Big) + \frac{(j\omega\varepsilon)^2}{2} a(x,\tau)\Phi(\omega,\tau).
\end{aligned} \tag{23}$$

In the system (23), terms containing C are destroyed, then we obtain:

$$\begin{aligned}
\frac{\partial \Phi(\omega,\tau)}{\partial \tau} =& (j\omega)^2\Phi(\omega,\tau)\Big((\lambda+\gamma_2)\Big(g_1 - \varphi_1\frac{\partial \Phi(\omega,\tau)/\partial\omega}{\omega\Phi(\omega,\tau)}\Big) + \\
&+ \lambda\Big(g_2 - \varphi_2\frac{\partial \Phi(\omega,\tau)/\partial\omega}{\omega\Phi(\omega,\tau)}\Big) - x(\tau)\Big(g_0 - \varphi_0\frac{\partial \Phi(\omega,\tau)/\partial\omega}{\omega\Phi(\omega,\tau)}\Big) + \\
&+ \frac{\partial \Phi(\omega,\tau)/\partial\omega}{\omega\Phi(\omega,\tau)} R_0\Big) + \frac{(j\omega\varepsilon)^2}{2} a(x,\tau)\Phi(\omega,\tau).
\end{aligned} \tag{24}$$

Let us rewrite the equation (24) by collecting identical terms.

$$\begin{aligned}
\frac{\partial \Phi(\omega,\tau)}{\partial \tau} =& -(j\omega)^2\Phi(\omega,\tau)\frac{\partial \Phi(\omega,\tau)/\partial\omega}{\omega\Phi(\omega,\tau)}\Big((\lambda+\gamma_2)\varphi_1 + \lambda\varphi_2 - \\
&- x(\tau)\varphi_0 - R_0\Big) + (j\omega)^2\Phi(\omega,\tau)\Big((\lambda+\gamma_2)g_1 + \lambda g_2 - x(\tau)g_0\Big) + \\
&+ \frac{(j\omega\varepsilon)^2}{2} a(x,\tau)\Phi(\omega,\tau).
\end{aligned} \tag{25}$$

Let us pay attention to the multiplier in the first term of the equation (25), then we get:

$$\begin{aligned}
-x(\tau)\varphi_0 + (\lambda+\gamma_2)\varphi_1 + \lambda\varphi_2 - R_0 =& -x(\tau)\frac{\partial R_0(x)}{\partial x} + (\lambda+\gamma_2)\frac{\partial R_1(x)}{\partial x} + \\
&+ \lambda\frac{\partial R_2(x)}{\partial x} - R_0(x) = a'(x).
\end{aligned}$$

Then the equation (25) will take the form:

$$\begin{aligned}
\frac{\partial \Phi(\omega,\tau)}{\partial \tau} =& \omega\frac{\partial \Phi(\omega,\tau)/\partial\omega}{\Phi(\omega,\tau)} a'(x) + \frac{(j\omega\varepsilon)^2}{2}\Phi(\omega,\tau)(a(x,\tau) + 2((\lambda+\gamma_2)g_1 + \\
&+ \lambda g_2 - x(\tau)g_0)).
\end{aligned}$$

Let us denote

$$b(x) = a(x,\tau) + 2\big((\lambda+\gamma_2)g_1(x) + \lambda g_2(x) - x(\tau)g_0(x)\big),$$

we obtain:

$$\frac{\partial \Phi(\omega,\tau)}{\partial \tau} = a'(x)\omega \frac{\partial \Phi(\omega,\tau)/\partial\omega}{\Phi(\omega,\tau)} + b(x)\frac{(j\omega\varepsilon)^2}{2}\Phi(\omega,\tau). \quad (26)$$

In this equation, function $b(x)$ is the diffusion coefficient of the diffusion process for which the transfer coefficient is function $a(x)$.

6. Construction of diffuse approximation

Next, applying the inverse Fourier transform to the equation (26), we move on to the equation for the probability density.

Taking into account the following ratios:

$$\begin{cases} \omega \dfrac{\partial \Phi(\omega,\tau)}{\partial \omega} = -\displaystyle\int_{-\infty}^{\infty} e^{j\omega y}\big(yP(y,\tau)\big)' dy, \\ \dfrac{(j\omega\varepsilon)^2}{2}\Phi(\omega,\tau) = \displaystyle\int_{-\infty}^{\infty} e^{j\omega y}\frac{\partial^2 P(y,\tau)}{\partial y^2} dy, \end{cases}$$

we obtain an equation that is the Fokker–Planck equation for the probability density of some diffusion process $y(\tau)$ with the transfer coefficient $a'(x)$ and the diffusion coefficient $b(x)$.

Thus, the process $y(\tau)$ is a solution to the stochastic differential equation

$$dy(z) = a'(x)y(\tau)d\tau + \sqrt{b(x)}d\omega(\tau),$$

where $\omega(\tau)$ is the Wiener process.

Let us introduce the diffusion process

$$z(\tau) = x(\tau) + \varepsilon y(\tau),$$

where the function $x(\tau)$ is a solution to the ordinary differential equation

$$dx(\tau) = a(x)d\tau.$$

Then the diffusion process $z(\tau)$ is a solution to the stochastic differential equation:

$$dz(\tau) = (a(x) + \varepsilon a'(x)y(\tau))d\tau + \varepsilon\sqrt{b(x)}d\omega(\tau).$$

Let us write the terms on the right side of the equation

$$a(x) + \varepsilon a'(x)y(\tau) = a(x+\varepsilon y) + O(\varepsilon^2) = a(z) + O(\varepsilon^2),$$

$$\varepsilon\sqrt{b(x)} = \varepsilon\sqrt{b(x+\varepsilon y - \varepsilon y)} = \varepsilon\sqrt{b(z-\varepsilon y)} = \varepsilon\sqrt{b(z)} + O(\varepsilon^2).$$

We will assume that terms of order of smallness greater than e do not make a significant contribution to the solution, which means we can neglect them. Then we get an equation of the form:

$$dz(\tau) = a(z)d\tau + \varepsilon\sqrt{b(z)}d\omega(\tau).$$

Let us denote the probability density of the diffusion process $z(\tau)$:

$$\Pi(z,\tau) = \frac{\partial\big(z(\tau) < z\big)}{\partial z}.$$

Let us write the Fokker–Planck equation for the diffusion process $z(\tau)$:

$$\frac{\partial \Pi(z,\tau)}{\partial \tau} = -\frac{\partial\big(a(z)\Pi(z,\tau)\big)}{\partial z} + \frac{\varepsilon^2}{2}\frac{\partial^2\big(b(z)\Pi(z,\tau)\big)}{\partial^2 z}. \tag{27}$$

In the equation (27) we make the reverse substitution $\sigma = \varepsilon^2$ and move on to the equation for the stationary probability distribution of the diffusion process $z(\tau)$:

$$\begin{gathered} -\big(a(z)\Pi(z)\big)' + \frac{\sigma}{2}\big(b(z)\Pi(z)\big)'' = 0, \\ \big(b(z)\Pi(z)\big)' = \frac{2}{\sigma}a(z)\Pi(z). \end{gathered} \tag{28}$$

To solve the equation (28), we introduce the substitution $G'(z) = b(z)\Pi(z)$, then we get:

$$G'(z) = \frac{2}{\sigma}\frac{a(z)}{b(z)}G(z), \tag{29}$$

where $a(z)$, $b(z)$ are the transfer and diffusion coefficients.

In the equation (29) we make the reverse substitution $\frac{G(z)}{b(z)} = \Pi(z)$, then the stationary probability density of the approximating random process has the form:

$$\Pi(z) = \frac{C}{b(z)}\exp\left(\frac{2}{\sigma}\int\limits_0^z \frac{a(z)}{b(x)}dx\right),$$

where C is a normalizing constant.

Let's construct a diffusion approximation using the formula:

$$PD(i) = \frac{\Pi(i\sigma)}{\sum\limits_{n=0}^{N}\Pi(n\sigma)}.$$

7. Results and discussion

We consider a system with parameters: $\lambda = 1$, $\mu_2 = 3$, $\gamma_1 = 0.1$, $\gamma_2 = 0.1$ and different system occupancy parameters $\rho = \frac{\lambda}{\mu_1}$.

Let us determine the accuracy of the approximation using the Kolmogorov distance

$$\begin{gathered} \Delta_1 = \max_{0\le i\le N}\left|\sum_{i=0}^{N} P\_\text{matrix}(i) - \sum_{i=0}^{N} P\_\text{diffusion}(i)\right|, \\ \Delta_2 = \max_{0\le i\le N}\left|\sum_{i=0}^{N} P\_\text{matrix}(i) - \sum_{i=0}^{N} P\_\text{asimpt}(i)\right|, \end{gathered}$$

where $P\_\text{matrix}(i)$ is the distribution obtained by the matrix method, $P\_\text{diffusion}(i)$ is the distribution obtained by the asymptotic-diffusion method and $P\_\text{asimpt}(i)$ is the distribution obtained by the asymptotic method.

As a condition for the applicability of the asymptotic-diffusion method, we take the threshold value of the Kolmogorov distance $\Delta = 0.05$.

Table 1 shows the accuracy between the matrix and asymptotic-diffusion distributions for various parameters and different system loads.

Similarly, Table 2 shows the accuracy between the matrix and asymptotic distributions.

Table 1

Kolmogorov distance

Δ_1	$\sigma = 2$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0.2$	$\sigma = 0.1$	$\sigma = 0.05$
$\rho = 0.6$	0.086	0.088	0.064	0.063	0.061	0.060
$\rho = 0.7$	0.062	**0.049**	**0.047**	**0.045**	**0.043**	**0.041**
$\rho = 0.8$	**0.035**	**0.031**	**0.032**	**0.028**	**0.026**	**0.020**
$\rho = 0.9$	**0.012**	**0.0097**	**0.019**	**0.014**	**0.0085**	**0.0013**

Table 2

Kolmogorov distance

Δ_2	$\sigma = 2$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0.2$	$\sigma = 0.1$	$\sigma = 0.05$
$\rho = 0.6$	0.160	0.124	0.100	0.076	0.053	**0.049**
$\rho = 0.7$	0.295	0.250	0.219	0.186	0.168	0.134
$\rho = 0.8$	0.439	0.391	0.348	0.304	0.223	0.267
$\rho = 0.9$	0.442	0.381	0.327	0.193	0.255	0.184

According to the data in Table 1 and Table 2, we can conclude that the accuracy of the diffusion approximation increases as the system load factor decreases. The method is applicable when $\rho = 0.8$ for all parameters σ. The accuracy of the diffusion approximation exceeds the accuracy of the Gaussian approximation.

8. Conclusions

In this work, a study of the M/M/1 RQ system with an unreliable server was carried out using the method of asymptotically diffusion analysis. The stationary distribution of server states, the transfer coefficient and the diffusion coefficient are found. A diffusion approximation is constructed. The accuracy of the approximation is determined using the Kolmogorov distance between the distributions constructed by the asymptotic diffusion method and the matrix method. It was proved that the asymptotically diffusion analysis method is more accurate. It is shown that the accuracy of the diffusion approximation exceeds the accuracy of the Gaussian approximation.

Author Contributions: Conceptualization, N.Voronina and S.Rozhkova; methodology, N.Voronina and S.Rozhkova; software, N.Voronina; validation, N.Voronina and S.Rozhkova; formal analysis, N.Voronina; investigation, N.Voronina and S.Rozhkova; resources, N.Voronina; data curation, N.Voronina; writing—original draft preparation, N.Voronina; writing—review and editing, N.Voronina and S.Rozhkova. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data sharing is not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Gross, D., Shortle, J. F., Thompson, J. M. & Harris, C. M. *Fundamentals of queueing theory* (John wiley & sons, 2011).
2. G.Gosztony. Repeated call attempts and their effect on trafic engineering. *Budavox Telecommunication Review* **2,** 16–26 (1976).
3. Cohen, J. Basic problems of telephone traffic theory and the influence of repeated calls. *Philips Telecommunication Rev.* **18,** 49 (1957).
4. Hanczewski, S., Stasiak, M., Weissenberg, J. & Zwierzykowski, P. *Queuing model of the access system in the packet network* in *Computer Networks* (2016), 283–293.
5. Falin, G. I. & Templeton, J. G. C. *Retrial Queues* 320 pp. doi:10.1201/9780203740767 (Chapman & Hall, London, 1997).
6. Artalejo, J. R. & Gómez-Corral, A. *Retrial Queueing Systems* doi:10.1007/978-3-540-78725-9 (Springer Berlin Heidelberg, 2008).
7. Dimitriou, I. A queueing model with two classes of retrial customers and paired services. *Annals of Operations Research* **238,** 123–143. doi:10.1007/s10479-015-2059-2 (2016).
8. Wang, J., Cao, J. & Li, Q. Reliability analysis of the retrial queue with server breakdowns and repairs. *Queueing Systems* **38,** 363–380 (2001).
9. Kumar, M. S. & Arumuganathan, R. An MX/G/1 retrial queue with two-phase service subject to active server breakdowns and two types of repair. *International Journal of Operational Research* **8,** 261–291 (2010).
10. Kim, C., Klimenok, V. I. & Orlovsky, D. S. The BMAP/PH/N retrial queue with Markovian flow of breakdowns. *European Journal of Operational Research* **189,** 1057–1072 (2008).
11. Lakaour, L., Aissani, D., Adel-Aissanou, K., Barkaoui, K. & Ziani, S. An unreliable single server retrial queue with collisions and transmission errors. *Communications in Statistics-Theory and Methods* **51,** 1085–1109 (2022).
12. Danilyuk, E. Y., Janos, S., *et al.* Asymptotic analysis of retrial queueing system M/M/1 with impatient customers, collisions and unreliable server. *Journal of Siberian Federal University. Mathematics & Physics* **13,** 218–230. doi:10.17516/1997-1397-2020-13-2-218-230 (2020).
13. Tóth, Á. & Sztrik, J. *Simulation of Finite-Source Retrial Queuing Systems With Collisions, Non-Reliable Server and Impatient Customers in the Orbit.* in *ICAI* (2020), 408–419.
14. Kuki, A., Bérczes, T., Sztrik, J. & Kvach, A. Numerical Analysis of Retrial Queueing Systems with Conflict of Customers and an Unreliable Server. *Journal of Mathematical Sciences* **237,** 673–683. doi:10.1007/s10958-019-04193-1 (2019).
15. Nazarov, A. A., Paul, S. V. & Lizyura, O. D. Two-way communication retrial queue with unreliable server and multiple types of outgoing calls. *Discrete and Continuous Models and Applied Computational Science* **28,** 49–61. doi:10.22363/2658-4670-2020-28-1-49-61 (2020).

16. Dudin, A., Dudina, O., Dudin, S. & Samouylov, K. Analysis of Single-Server Multi-Class Queue with Unreliable Service, Batch Correlated Arrivals, Customers Impatience, and Dynamical Change of Priorities. *Mathematics* **9,** 1257. doi:10.3390/math9111257 (2021).
17. Chakravarthy, S. R., OZKAR, S. & SHRUTI, S. Analysis of M/M/C retrial queue with thresholds, PH distribution of retrial times and unreliable servers. *Journal of applied mathematics & informatics* **39,** 173–196 (2021).
18. Falin, G. An M/G/1 retrial queue with an unreliable server and general repair times. *Performance Evaluation* **67,** 569–582 (2010).
19. Dudin, A., Klimenok, V. & Vishnevsky, V. *Analysis of unreliable single server queueing system with hot back-up server* in *Optimization in the Natural Sciences: 30th Euro Mini-Conference, EmC-ONS 2014, Aveiro, Portugal, February 5-9, 2014. Revised Selected Papers 30* (2015), 149–161.
20. Nazarov, A. A., Paul, S. V., Lizyura, O. D., *et al.* Two-way communication retrial queue with unreliable server and multiple types of outgoing calls. *Discrete and Continuous Models and Applied Computational Science.* doi:10.22363/2658-4670-2020-28-1-49-61 (2020).
21. Voronina, N. M., Fedorova, E. A. & Rozhkova, S. V. Asymptotic analysis of the RQ-system M/M/1 with an unreliable server. *Mathematical and software for information technical and economic systems,* 304–309. doi:10.1007/978-3-031-09331-9_28 (2020).

Information about the authors

Nataliya M. Voronina—Senior Lecturer of Department of Information Technology of School of Information Technology and Robotics Engineering, National Research Tomsk Polytechnic University (e-mail: vnm@tpu.ru, phone: +7 (3822) 701-777-11-54, ORCID: 0000-0001-9044-5211, ResearcherID: AAD-2035-2019, Scopus Author ID: 57802914700)

Svetlana V. Rozhkova—Doctor of Physics and Mathematics Sciences, Professor of Department of Mathematics and Mathematical Physics of School of Nuclear Technology Engineering, National Research Tomsk Polytechnic University, Professor of Department of Probability Theory and Mathematical Statistics Institute of Applied Mathematics and Computer Science, National Research Tomsk State University (TSU) (e-mail: rozhkova@tpu.ru, phone: +7 (3822) 701-777-68-98, ORCID: 0000-0002-8888-9291, ResearcherID: F-5512-2017, Scopus Author ID: 6603581666)

УДК 519.872:519.217
PACS 07.05.Tp, 02.60.Pn, 02.70.Bf
DOI: 10.22363/2658-4670-2024-32-4-380-394 EDN: DDNAPR

Асимптотически диффузионный анализ RQ системы с ненадёжным прибором

Н.М. Воронина[1], С.В. Рожкова[1, 2]

[1] Национальный исследовательский Томский политехнический университет, пр. Ленина, д. 30, Томск, 634050, Российская Федерация

[2] Национальный исследовательский Томский государственный университет, пр. Ленина, д. 36, Томск, 634050, Российская Федерация

Аннотация. В работе рассматривается однолинейная RQ-система массового обслуживания с ненадёжным прибором. Системы массового обслуживания называются ненадёжными, если их приборы могут время от времени выходить из строя и требовать восстановления (ремонта), только после которого они могут возобновить обслуживание запросов. Исследование проводится методом асимптотически диффузионного анализа в условии большой задержки заявок на орбите. Найдены стационарное распределение состояний прибора, коэффициент переноса и коэффициент диффузии. Построена диффузионная аппроксимация. Доказано, что точность диффузионной аппроксимации превышает точность гауссовской аппроксимации.

Ключевые слова: RQ система, асимптотически диффузионный анализ, ненадежный прибор