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Numerical study of the φ^4 standing waves in a ball of finite radius

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Abstract. Study of spherically symmetric time-periodic standing waves of the ϕ^4 model in a ball of finite radius was carried out based on the numerical solution of a boundary value problem on a cylindrical surface for a wide range of values of the oscillation period. The standing waves in a ball of finite radius can be considered as an approximation of weakly radiating spherically symmetric oscillons in the ϕ^4 model. Stability analysis the waves obtained is based on the calculation of the corresponding Floquet multipliers. In the paper, mathematical formulation of the problem is given, the numerical approach is described, including the method of parallel implementation of the calculation of Floquet multipliers on the computing resources of the HybriLIT platform of the Multifunctional Information and Computing Complex of the Joint Institute for Nuclear Research (Dubna). The results of the study of the space-time structure and bifurcation of coexisting standing waves of various types are presented.

Key words and phrases: oscillons, numerical study, parallel computing

1. Introduction

We consider spherically symmetric standing waves in the ϕ^4 equation

$$\Phi_{tt} - \Delta\Phi - \Phi + \Phi^3 = 0, \quad \Delta = \frac{d^2}{dr^2} + \frac{2}{r}. \tag{1}$$

Equation (1) can be used as a model for a wide range of nonlinear wave processes within various physical contexts. The localized long-lived pulsating states (pulsons, oscillons) in the three-dimensional ϕ^4 theory are known since 1975 [1]. Subsequent computer simulations [2, 3] revealed the formation of long-lived pulsating structures of large amplitude and nearly unchanging width, see a fragment of such structure in figure 1. Renewed interest to oscillons is inspired by applications in cosmology and high energy physics (see, i.e. [4–7]).

Due the permanent loss of energy to the second-harmonic radiation, the oscillons are not exactly time-periodic. In contrast, the standing waves are periodic and can be determined as solutions of a boundary-value problem on the cylindrical surface. In a recent paper [8], the standing waves in a ball of large finite radius are considered as approximation of infinite-space weakly radiating spherically symmetric oscillons in the ϕ^4 model. In the present contribution, we provide details of the numerical approach and present results on the spatio-temporal structure and bifurcation of the standing waves. We outline the numerical scheme that was utilised for that purpose as well as its parallel computer implementation.

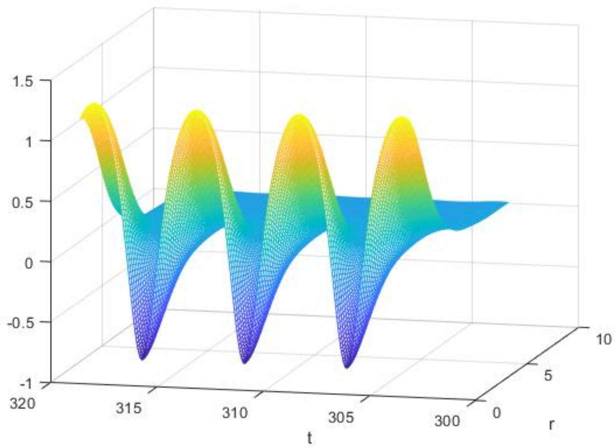

Figure 1. A fragment of pulsating radial component of spherically symmetric solution of Eq. (1)

2. Mathematical problem and numerical approach

Let $\phi = \Phi - \Phi_0$ where $\Phi(r,t)$ is a spherically symmetric solution of equation (1) approaching $\Phi_0 = -1$ (one of two vacuum solutions) as $r \to \infty$. Thus, equation (1) takes a form:

$$\phi_{tt} - \phi_{rr} - \frac{2}{r}\phi_r + 2\phi - 3\phi^2 + \phi^3 = 0. \tag{2}$$

We are searching for solutions of the equation (2) satisfying the boundary conditions

$$\phi_r(0,t) = \phi(R,t) = 0, \quad \phi(r,T) = \phi(r,0). \tag{3}$$

Letting $\tau = t/T$ and defining $\psi(r,\tau) = \phi(r,t)$ yields

$$\psi_\tau(r,\tau) = T\phi_t(r,t), \quad \psi_{\tau\tau}(r,\tau) = T^2\phi_{tt}(r,t).$$

Thus, a boundary-value problem in the two-dimensional domain $[0,1] \times [0,R]$ takes a form:

$$\psi_{tt} + T^2 \cdot \left[-\psi_{rr} - \frac{2}{r}\psi_r + 2\psi - 3\psi^2 + \psi^3\right] = 0, \tag{4}$$

$$\psi_r(0,t) = \psi(R,t) = 0, \quad \psi(r,1) = \psi(r,0). \tag{5}$$

For each value of T the boundary-value problem (4),(5) was solved by means of the Newtonian iteration [9] with the 2nd order finite difference approximation for the derivatives. The t and r discrete steps were taken to be 0.01 and 0.1, respectively.

Any solution of equation (4) can be characterised by its energy

$$E = 4\pi \int_0^R \left(\frac{\phi_t^2}{2} + \frac{\phi_r^2}{2} + \phi^2 - \phi^3 + \frac{\phi^4}{4}\right) r^2 dr \tag{6}$$

and its corresponding frequency $\omega = \frac{2\pi}{T}$. If the solution with frequency ω does not change appreciably as R is increased — in particular, if the energy (6) does not change — this standing wave provides a fairly accurate approximation for the periodic solution in an infinite space.

Solutions of equation (4) were numerically continued in the parameter T and the energy-frequency diagram was constructed. Numerical continuation was organized as described in [10].

To classify the stability of the resulting standing waves against spherically symmetric perturbations we consider the linearised equation

$$y_{tt} - y_{rr} - \frac{2}{r} y_r - y + 3(\phi - 1)^2 y = 0 \tag{7}$$

with the boundary conditions $y_r(0,t) = y(R,t) = 0$ [8]. Expanding $y(r,t)$ in Fourier sine series produces a system of $2N$ ordinary differential equations for the coefficients:

$$\dot{u}_m = v_m, \qquad \dot{v}_m + F = 0. \tag{8}$$

Here

$$F = (2 + k_m^2) u_m - 3 \sum_{n=1}^{N} (A_{m-n} - A_{m+n}) u_n + \frac{3}{2} \sum_{n=1}^{N} (A_{m-n} - A_{m+n}) u_n,$$

and A_n, B_n are T-periodic functions of t:

$$A_n(t) = \frac{2}{R} \int_0^R \phi(r,t) \cos(k_n r) dr, \;\; B_n(t) = \frac{2}{R} \int_0^R \phi^2(r,t) \cos(k_n r) dr.$$

A set of $2N$ linearly independent solutions, evaluated numerically at $t = T$, forms a monodromy matrix of the system (8). The standing-wave solution $\phi(r,t)$ is deemed stable if all of its $2N$ eigenvalues μ of the monodromy matrix (Floquet multipliers) lie on the unit circle and unstable if there are multipliers outside the circle, see example of stability and instability case at the figure 2.

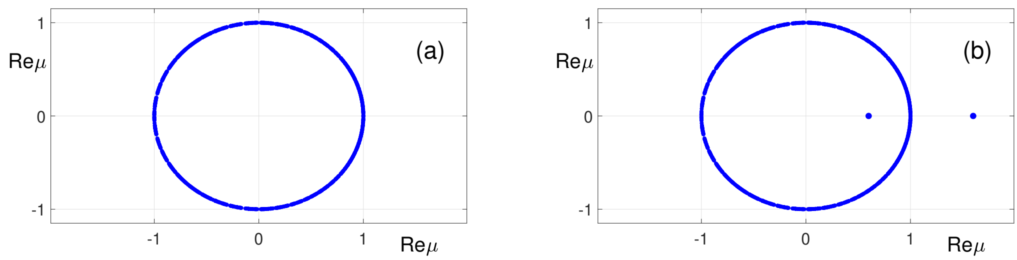

Figure 2. Floquet multipliers at the (Reμ,Imμ) plane. Stability case (a) T=4.7206; instability case (b): T=5.025. Here R=100

The above numerical approach was implemented using the MATLAB *ode45* function with the tolerance value 10^{-7}. A cubic spline interpolation was employed for the calculation of the $A_{m\pm n}(t)$ and $B_{m\pm n}(t)$ coefficients at a set of t points. With N=1000 Fourier harmonics, the calculation of the Floquet multipliers for each individual value of T takes about 48 hours and 24 hours on the *HybriLIT* cluster and the *Govorun* supercomputer, respectively.

To speed up the computations, a parallel algorithm was implemented based on the *parfor* operator. This produces an automated splitting of the solution of $2N$ Cauchy problems into available parallel threads, or "workers". The speedup of calculations in parallel mode can reach 20 times compared to the single- thread calculations.

3. Numerical results and conclusions

It was pointed out in [8] that the energy-frequency dependence, $E(\omega)$, obtained by a numerical continuation of solutions of equation (4) looks like a sequence of spikes triggered by the resonance of frequencies of two coexisting solutions. Positions of the spikes are R-sensitive. It can be seen in figure 3 where fragments of this diagram for R=100 and R=150 are shown.

The numerical study shows that the boundary value problem (4),(5) has a set of two coexisting spherically symmetric standing wave solutions. They are shown in figure 4(a,b). The first one is the Bessel-like wave without an explicitly localised in space core decaying in proportion to r^{-1} as $r \to R$) that branches off the zero solution, see figure 4a. The second type wave is characterised by an exponentially localised in space and pulsating in time core with a small- amplitude slowly decaying in space second-harmonic tail as shown in figure 4b. The corresponding curves $\phi(r,0)$ for both waves are shown in figure 4c. Both solutions are shown at the point marked by the arrow in figure 4d: $\omega = 0.9802\,\omega_0$, E=45.585. Figure 4d demonstrates structure of a resonance spike and interconnection between the $E(\omega/\omega_0)$ branches of two types of waves. The Bessel-like branch extends to zero energy (red dashed curve in figure 4d). Both slopes of the resonance spike (blue solid curves in figure 4d) join the Bessel-like branch at points of period-doubling bifurcations.

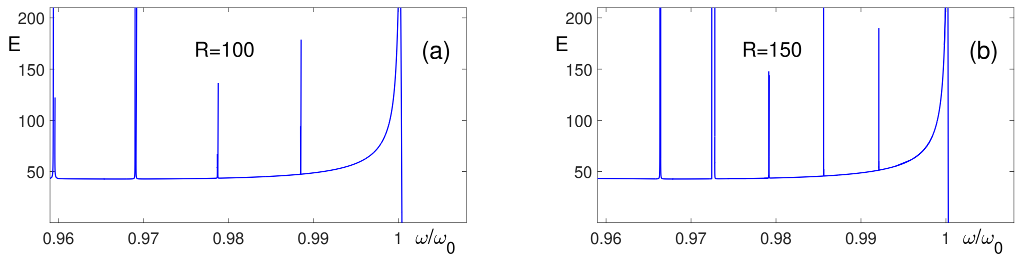

Figure 3. The energy-frequency diagram produced by numerical continuation of solutions of the boundary value problem (4), (5) for the cases R=100 (a) and R=150 (b)

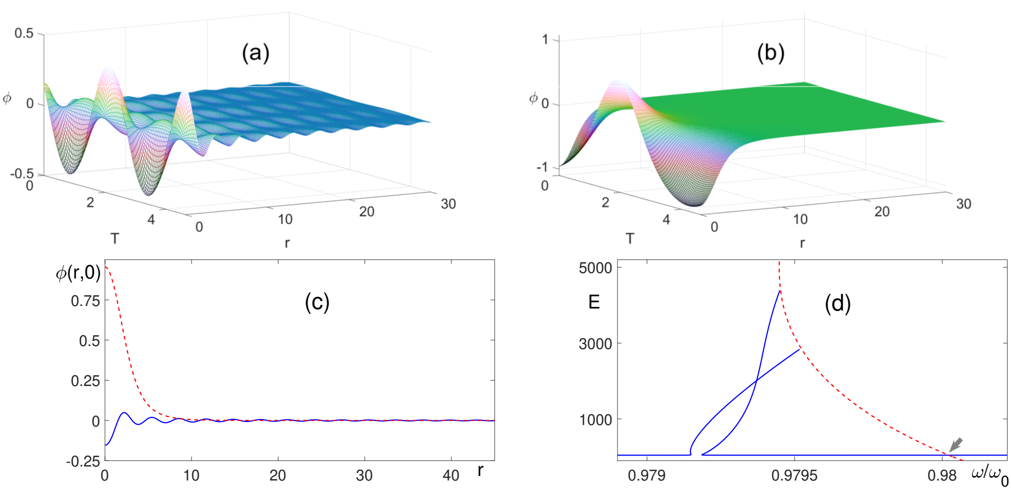

Figure 4. Coexisting standing waves in a ball of radius R=150 at the point $\omega = 0.9802\,\omega_0$: the Bessel-like wave (a) and the wave with exponentially localised pulsating core (b). (c): Curves $\phi(0,r)$ for the waves of the first (blue solid) and the second type (red dashed). (d): The $E(\omega/\omega_0)$ structure of the resonance spike region. The red dashed curve corresponds the Bessel-like wave. The arrow marks the point where coexisting waves (a) and (b) are shown

Stability analysis of the Bessel waves against the spherically symmetric perturbations shows that the Bessel waves are stable for low energy values starting from E=0 up to the bifurcation point at their intersection with either slope of the resonance spike. As for the second type of exponentially localised standing waves, the calculations in [8] show they have only short stability intervals between ω=0.9428ω_0 and ω=0.9435ω_0 inside the ball of radius R=40. In the case of a ball of larger radius, there are sections of stable frequencies of greater width. Note that the Bessel-like curve in figure 4d (red dashed) is stable for the values of E below the intersection with the left slope of the spike at the point $\omega = 0.97952\,\omega_0$, E=2879.3. All other solutions in figure 4d are found to be unstable.

It is important to emphasize that the curve $E(\omega)$ has a single minimum at $\omega_{\min} = 0.967\omega_0$ for all values of the radius R despite the fact that the number and the positions of the spikes are R-sensitive. Thus, we can consider the energy curve at least in the neighbourhoods away from the spikes to be the approximation of the radius-independent envelope of the nearly-periodic oscillons in the infinite space.

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Численное исследование стоячих волн модели φ^4 в шаре конечного радиуса

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Аннотация. Проведено численное исследование сферически симметричных периодических по времени стоячих волн модели ϕ^4 в шаре конечного радиуса на основе вычисления решений сформулированной нелинейной краевой задачи на цилиндрической поверхности в широком диапазоне значений периода осцилляций и последующего анализа устойчивости полученных таким образом решений путем расчета соответствующих множителей Флоке. При этом стоячие волны в шаре конечного радиуса могут рассматриваться как аппроксимацияя слабоизлучающих сферически-симметричных осциллонов в модели ϕ^4. В работе описывается математическая постановка задачи и метод ее численного решения, обсуждается метод параллельной реализации расчета множителей Флоке на вычислительных ресурсах платформы HybriLIT Многофункционального информационно-вычислительного комплекса Объединенного института ядерных исследований (Дубна). Представлены результаты по исследованию пространственно-временной структуры и бифуркации сосуществующих стоячих волн различного типа.

Ключевые слова: осциллоны, компьютерное моделирование, параллельные вычисления