Discrete & Continuous Models
& Applied Computational Science
ISSN 2658-7149 (online), 2658-4670 (print)

2023, 31 (4) 399–418
http://journals.rudn.ru/miph

Research article

UDC 537.8:514.762.37
DOI: 10.22363/2658-4670-2023-31-4-399-418
EDN: GCUXWK

Methodological derivation of the eikonal equation

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(received: December 1, 2023; revised: December 19, 2023; accepted: December 29, 2023)

Abstract. Usually, when working with the eikonal equation, reference is made to its derivation in the monograph by Born and Wolf. The derivation of this equation was done rather carelessly. Understanding this derivation requires a certain number of implicit assumptions. For a better understanding of the eikonal approximation and for methodological purposes, the authors decided to repeat the derivation of the eikonal equation, explicating all possible assumptions. Methodically, the following algorithm for deriving the eikonal equation is proposed. The wave equation is derived from Maxwell's equation. In this case, all conditions are explicitly introduced under which it is possible to do this. Further, from the wave equation, the transition to the Helmholtz equation is carried out. From the Helmholtz equation, with the application of certain assumptions, a transition is made to the eikonal equation. After analyzing all the assumptions and steps, the transition from the Maxwell's equations to the eikonal equation is actually implemented. When deriving the eikonal equation, several formalisms are used. The standard formalism of vector analysis is used as the first formalism. Maxwell's equations and the eikonal equation are written as three-dimensional vectors. After that, both the Maxwell's equations and the eikonal equation use the covariant 4-dimensional formalism. The result of the work is a methodically consistent description of the eikonal equation.

Key words and phrases: eikonal, Maxwell's equations, wave equation, vector representation, tensor representation

1. Introduction

One of the foundations of the simulation program we employ for modeling optical phenomena is the eikonal model [1, 2]. While this model is well-known, the derivation process is somewhat intricate [3, 4]. In the renowned monograph by Born and Wolf [5], the derivation appears almost like a form of physical magic (here are Maxwell's equations, a bit of magic, and voila, we have the eikonal equation). We decided to delve deeper into the derivation of the eikonal equation.

We employed analytical methods to derive the eikonal equation from Maxwell's equations in a medium without currents and charges. The process involves analyzing differential equations and applying methods of mathematical analysis. A brief outline of our study is presented in the scheme shown in the figure 1.

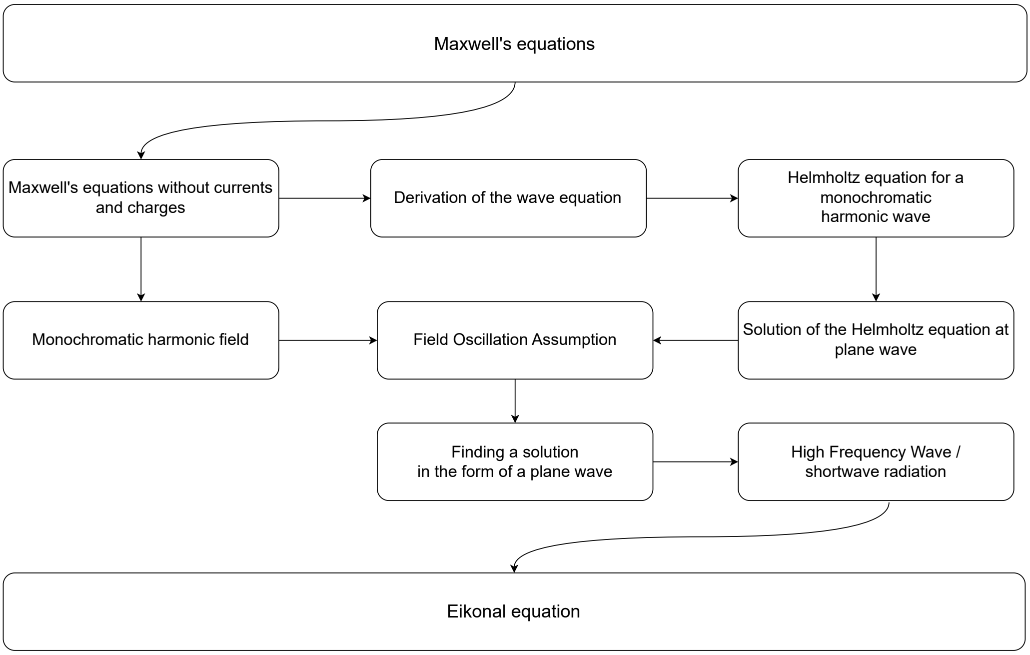

Figure 1. Paper structure

1.1. Article structure

In section 1.2, we present basic notations and conventions used in the article. In section 2, fundamental relationships for Maxwell's equations are introduced. In section 3, the wave equation is derived from Maxwell's equations. Next, in section 4, the eikonal equation is obtained from the wave equation. The transformations are performed using vector formalism. In section 5, the same is done based on covariant tensor formalism.

1.2. Notations and conventions

1. The primary mathematical framework used in the article is the vector analysis (a brief overview is given in Appendix) and tensor analysis.

2. We will adhere to the following conventions. Greek indices (α, β) will refer to the four-dimensional space, with the component values as follows: $\alpha = \overline{0,3}$. Latin indices from the middle of the alphabet (i, j, k) will refer to the three-dimensional space, with the component values as follows: $i = \overline{1,3}$.
3. The CGS symmetrical system [6] is used for notating the equations of electrodynamics.

2. Introduction

Consider Maxwell's equations in vector-differential form:

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c}\mathbf{j}, \tag{1}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \tag{2}$$

$$\nabla \cdot \mathbf{D} = 4\pi\rho, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

where
— $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(x, y, z, t)$ is electric field strength vector;
— $\mathbf{H}(\mathbf{r}, t) = \mathbf{H}(x, y, z, t)$ is magnetic field strength vector;
— $\mathbf{D}(\mathbf{r}, t) = \mathbf{D}(x, y, z, t)$ is electric field induction vector;
— $\mathbf{B}(\mathbf{r}, t) = \mathbf{B}(x, y, z, t)$ is magnetic field induction vector;
— $\mathbf{j}(\mathbf{r}) = \mathbf{j}(x, y, z)$ is external electric current density (current strength per unit area);
— $\rho(\mathbf{r}) = \rho(x, y, z)$ is electric charge density;
— c is vacuum speed of light;
— $\mathbf{r} = (x, y, z)^T$ is radius vector of a point, written in Cartesian coordinates.

Let us briefly describe the physical meaning of each of Maxwell's equations:
— equation (1) means that electric current and a change in electric induction generate a solenoidal magnetic field, that is a field whose field lines twist into a vortex along the vector indicating the direction of the current;
— equation (2) means that a change in time of a magnetic field generates an electric field;
— equation (3) means that the electric charge is the source of electrical induction;
— equation (4) means that there are no free magnetic poles (only magnetic dipoles have been experimentally discovered, magnetic monopoles are not known to science).

The following relations, called *material equations*, are also valid:

$$\mathbf{j} = \sigma\mathbf{E}, \quad \mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H},$$

where $\sigma(\mathbf{r})$ is conductivity, $\varepsilon(\mathbf{r})$ is permittivity, and $\mu(\mathbf{r})$ is permeability. In an isotropic medium, ε and μ are scalar quantities, but in the general case they are tensor quantities.

A medium is called *isotropic* if its physical properties do not depend on direction. The term comes from the Greek words «izos» (ισος): equal, identical, similar and "tropos" (τροπος): direction, character. In electrodynamics, the isotropy of a medium is associated with the same values of $\varepsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ in all directions.

Magnetic permeability characterizes the magnetic properties of a medium (substance). If $\mu \neq 1$, then the substance is called *magnetic*, if $\mu > 1$ — *paramagnetic*, if $\mu < 1$ — *diamagnetic*.

Next, we will consider a medium that does not conduct electricity, that is, $\sigma = 0$, and also free from currents, that is, $\mathbf{j} = \mathbf{0}$ and $\rho = 0$, then Maxwell's equations simplified somewhat:

$$\boldsymbol{\nabla} \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \mathbf{0}, \tag{5}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \tag{6}$$

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 0, \tag{7}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{8}$$

3. Wave equation

3.1. Derivation of the wave equation from Maxwell's equations

We will assume that $\rho = 0$ and $\mathbf{j} = \mathbf{0}$ and consider the equations (5) and (6):

$$\boldsymbol{\nabla} \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \mathbf{0},$$
$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}.$$

We use the material equations $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ and take into account the dependence of the permittivity and permeability on coordinates: $\varepsilon = \varepsilon(x, y, z)$ and $\mu = \mu(x, y, z)$.

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}(\mu\mathbf{H}) = \mathbf{0} \Rightarrow \boldsymbol{\nabla} \times \mathbf{E} + \frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t} = \mathbf{0} \Rightarrow \frac{1}{\mu}\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = \mathbf{0}.$$

Apply the curl operator $\boldsymbol{\nabla}\times$ to the resulting equation:

$$\underbrace{\boldsymbol{\nabla} \times \left(\frac{1}{\mu}\boldsymbol{\nabla} \times \mathbf{E}\right)}_{\text{(I)}} + \underbrace{\frac{1}{c}\boldsymbol{\nabla} \times \frac{\partial \mathbf{H}}{\partial t}}_{\text{(II)}} = \mathbf{0}.$$

Let us first consider term (II) of this equation. The time derivative can be taken out from under the sign of the rotor operator:

$$\boldsymbol{\nabla} \times \frac{\partial \mathbf{H}}{\partial t} = \frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{H}).$$

Due to (5) we get:

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{H})=\frac{\partial}{\partial t}\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}=\frac{1}{c}\frac{\partial^2\mathbf{D}}{\partial t^2}.$$

We use the material equation $\mathbf{D}=\varepsilon\mathbf{E}$ and write as follows:

$$\frac{\partial^2\mathbf{D}}{\partial t^2}=\frac{\partial^2\left(\varepsilon(\mathbf{r})\mathbf{E}\right)}{\partial t^2}=\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}\Rightarrow\frac{1}{c}\boldsymbol{\nabla}\times\frac{\partial\mathbf{H}}{\partial t}=\frac{\varepsilon}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

To simplify term (I), we use the relation $\boldsymbol{\nabla}\times f\mathbf{v}=f\boldsymbol{\nabla}\times\mathbf{v}+\boldsymbol{\nabla}f\times\mathbf{v}$, where $f(x,y,z)$ is a scalar function, and $\mathbf{v}(x,y,z)$ is a vector field. Using this relation, term (I) is expanded as follows:

$$\boldsymbol{\nabla}\times\left(\frac{1}{\mu}\boldsymbol{\nabla}\times\mathbf{E}\right)=\underbrace{\frac{1}{\mu}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E})}_{\text{(I.a)}}+\underbrace{\left(\boldsymbol{\nabla}\frac{1}{\mu},\boldsymbol{\nabla}\times\mathbf{E}\right)}_{\text{(I.b)}}.$$

In turn, to simplify term (I.a) we use the identity $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{v}=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v})-\nabla^2\mathbf{v}$, where ∇^2 is the Laplace operator.

$$\frac{1}{\mu}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E})=\frac{1}{\mu}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E})-\frac{1}{\mu}\nabla^2\mathbf{E}.$$

To simplify the expression $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E})$ we apply the identity $\boldsymbol{\nabla}\cdot(f\mathbf{v})=f\boldsymbol{\nabla}\cdot\mathbf{v}+(\boldsymbol{\nabla}f,\mathbf{v})$ to Maxwell's equation (7), replacing induction $\mathbf{D}$ with tension using the material equation $\mathbf{D}=\varepsilon\mathbf{E}$:

$$\begin{aligned}&\boldsymbol{\nabla}\cdot\mathbf{D}=\boldsymbol{\nabla}\cdot(\varepsilon\mathbf{E})=\varepsilon\boldsymbol{\nabla}\cdot\mathbf{E}+(\mathbf{E},\boldsymbol{\nabla}\varepsilon)=0\Rightarrow\\&\qquad\Rightarrow\boldsymbol{\nabla}\cdot\mathbf{E}=-\frac{1}{\varepsilon}\left(\boldsymbol{\nabla}\varepsilon,\mathbf{E}\right)\Rightarrow\frac{1}{\mu}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E})=-\frac{1}{\mu}\boldsymbol{\nabla}\left(\frac{1}{\varepsilon}(\boldsymbol{\nabla}\varepsilon,\mathbf{E})\right).\end{aligned}$$

As a result, the term (I.a) was transformed to the following form:

$$\frac{1}{\mu}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E})=-\frac{1}{\mu}\boldsymbol{\nabla}\left(\frac{1}{\varepsilon}(\boldsymbol{\nabla}\varepsilon,\mathbf{E})\right)-\frac{1}{\mu}\nabla^2\mathbf{E}.$$

By combining (I) and (II) we write:

$$\underbrace{-\frac{1}{\mu}\nabla^2\mathbf{E}-\frac{1}{\mu}\boldsymbol{\nabla}\left(\frac{1}{\varepsilon}(\boldsymbol{\nabla}\varepsilon,\mathbf{E})\right)}_{\text{(I.a)}}+\underbrace{\left(\boldsymbol{\nabla}\frac{1}{\mu},\boldsymbol{\nabla}\times\mathbf{E}\right)}_{\text{(I.b)}}+\underbrace{\frac{\varepsilon}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}}_{\text{(II)}}=0,$$

$$-\nabla^2\mathbf{E}-\boldsymbol{\nabla}\left(\frac{1}{\varepsilon}(\boldsymbol{\nabla}\varepsilon,\mathbf{E})\right)+\mu\left(\boldsymbol{\nabla}\frac{1}{\mu},\boldsymbol{\nabla}\times\mathbf{E}\right)+\frac{\mu\varepsilon}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0,$$

$$\nabla^2\mathbf{E}-\frac{\mu\varepsilon}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}+\left[\boldsymbol{\nabla}\left(\frac{1}{\varepsilon}(\boldsymbol{\nabla}\varepsilon,\mathbf{E})\right)-\mu\left(\boldsymbol{\nabla}\frac{1}{\mu},\boldsymbol{\nabla}\times\mathbf{E}\right)\right]=0.$$

A completely similar equation can be obtained for the magnetic field strength vector $\mathbf{H}$.

In the case of an isotropic medium, that is, $\varepsilon = \mu = \text{const}$ the additional term taken in square brackets vanishes and we obtain the wave equation:

$$\nabla^2\mathbf{E} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \mathbf{0},$$
$$\nabla^2\mathbf{H} - \frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = \mathbf{0}.$$

We can introduce the quantity $v = c/\sqrt{\varepsilon\mu}$ — the speed of the electromagnetic wave in the medium.

3.2. The case of a plane wave

Consider the wave equation:

$$\nabla^2\mathbf{U} - \frac{1}{v^2}\frac{\partial^2\mathbf{U}}{\partial t^2} = 0.$$

Let's consider an electromagnetic wave that propagates in the direction $\mathbf{s}$, where $\mathbf{s} = (s_x, s_y, s_z)$ — some unit vector ($\|\mathbf{s}\| = 1$) fixed direction. Any solution of this equation, having the form $\mathbf{U} = \mathbf{U}((\mathbf{r}, \mathbf{s}), t)$ is a *plane* wave, since at every moment of time the vector $\mathbf{U}$ is constant in the plane $(\mathbf{r}, \mathbf{s}) = -d$, where $|d|$ is the distance from the plane to the origin. The expression $(\mathbf{r}, \mathbf{s}) = -d$ is actually a normal plane equation, where the vector $\mathbf{s}$ acts as the unit normal vector. Let's write it in Cartesian coordinates:

$$s_x x + s_y y + s_z z + d = 0.$$

The wave equation can be simplified by introducing a new coordinate system. Since the intensity vector of a plane wave entirely depends only on the distance d, we can choose a new coordinate system with axes $O\xi, O\eta, O\zeta$ so that the $O\zeta$ axis is directed along the vector $\mathbf{s}$, and the origin coincides with the previous Cartesian system $Oxyz$. Then, the coordinate along the $O\zeta$ axis will depend on the previous coordinates according to the formula $\zeta(x, y, z) = (\mathbf{r}, \mathbf{s}) = s_x x + s_y y + s_z z$, while ξ and η do not depend on the previous coordinates and can be chosen arbitrarily, for example, so that the coordinate system $O\xi\eta\zeta$ is right-handed (see the figure 2).

The replacement of differential operators is carried out using the Jacobian matrix as follows:

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial\xi}{\partial x} & \dfrac{\partial\eta}{\partial x} & \dfrac{\partial\zeta}{\partial x} \\ \dfrac{\partial\xi}{\partial y} & \dfrac{\partial\eta}{\partial y} & \dfrac{\partial\zeta}{\partial y} \\ \dfrac{\partial\xi}{\partial z} & \dfrac{\partial\eta}{\partial z} & \dfrac{\partial\zeta}{\partial z} \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \\ \dfrac{\partial}{\partial \zeta} \end{bmatrix}; \quad \left(\frac{\partial(\xi, \eta, \zeta)}{\partial(x, y, z)}\right)^T = \begin{bmatrix} \dfrac{\partial\xi}{\partial x} & \dfrac{\partial\eta}{\partial x} & \dfrac{\partial\zeta}{\partial x} \\ \dfrac{\partial\xi}{\partial y} & \dfrac{\partial\eta}{\partial y} & \dfrac{\partial\zeta}{\partial y} \\ \dfrac{\partial\xi}{\partial z} & \dfrac{\partial\eta}{\partial z} & \dfrac{\partial\zeta}{\partial z} \end{bmatrix}.$$

Since $\xi = \text{const}$ and $\eta = \text{const}$, and $\zeta = (\mathbf{r}, \mathbf{s})$, then:

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & s_x \\ 0 & 0 & s_y \\ 0 & 0 & s_z \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \\ \dfrac{\partial}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} s_x \dfrac{\partial}{\partial \zeta} \\ s_y \dfrac{\partial}{\partial \zeta} \\ s_z \dfrac{\partial}{\partial \zeta} \end{bmatrix} \Longrightarrow \begin{cases} \dfrac{\partial}{\partial x} = s_x \dfrac{\partial}{\partial \zeta}, \\ \dfrac{\partial}{\partial y} = s_y \dfrac{\partial}{\partial \zeta}, \\ \dfrac{\partial}{\partial z} = s_z \dfrac{\partial}{\partial \zeta}. \end{cases}$$

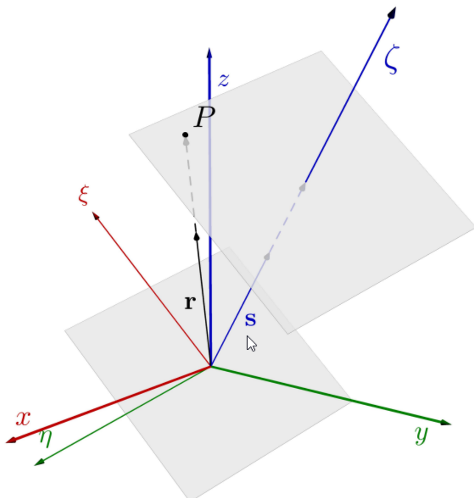

Figure 2. Plane $(\mathbf{r}, \mathbf{s}) = \text{const}$, where $\mathbf{s}$ is a unit vector indicating the direction of propagation of the electromagnetic wave. New coordinate axes are chosen so that the vector $\mathbf{s}$ is the unit vector of the $O\zeta$ axis. The other two axes $O\xi$ and $O\eta$ are chosen arbitrarily and form a right-handed coordinate system $O\xi\eta\zeta$

The Laplace operator after replacing coordinates is transformed to the following form:

$$\nabla^2 \mathbf{U} = s_x^2 \frac{\partial^2 \mathbf{U}}{\partial \zeta^2} + s_y^2 \frac{\partial^2 \mathbf{U}}{\partial \zeta^2} + s_z^2 \frac{\partial^2 \mathbf{U}}{\partial \zeta^2} = \left(s_x^2 + s_y^2 + s_z^2\right) \frac{\partial^2 \mathbf{U}}{\partial \zeta^2} = \frac{\partial^2 \mathbf{U}}{\partial \zeta^2}.$$

The wave equation simplifies:

$$\frac{\partial^2 \mathbf{U}}{\partial \zeta^2} - \frac{1}{v^2} \frac{\partial^2 \mathbf{U}}{\partial t^2} = 0.$$

We perform another substitution $p = \zeta - vt$ and $q = \zeta + vt$, which leads to the following transformation of the differential operators:

$$\begin{bmatrix} \dfrac{\partial}{\partial \zeta} \\ \dfrac{\partial}{\partial t} \end{bmatrix} = \left(\dfrac{\partial(\zeta, t)}{\partial(p, q)} \right)^T \begin{bmatrix} \dfrac{\partial}{\partial p} \\ \dfrac{\partial}{\partial q} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial p}{\partial \zeta} & \dfrac{\partial q}{\partial \zeta} \\ \dfrac{\partial p}{\partial t} & \dfrac{\partial q}{\partial t} \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial p} \\ \dfrac{\partial}{\partial q} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 \\ -v & v \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial p} \\ \dfrac{\partial}{\partial q} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial p} + \dfrac{\partial}{\partial q} \\ -v\dfrac{\partial}{\partial p} + v\dfrac{\partial}{\partial q} \end{bmatrix} \Rightarrow \begin{cases} \dfrac{\partial}{\partial \zeta} = \dfrac{\partial}{\partial p} + \dfrac{\partial}{\partial q}, \\ \dfrac{\partial}{\partial t} = -v\dfrac{\partial}{\partial p} + v\dfrac{\partial}{\partial q}. \end{cases}$$

The second derivatives are expressed through the new variables as follows:

$$\frac{\partial^2}{\partial \zeta^2} = \frac{\partial^2}{\partial p^2} + 2\frac{\partial}{\partial p}\frac{\partial}{\partial q} + \frac{\partial^2}{\partial q^2},$$
$$\frac{\partial^2}{\partial t^2} = v^2\left(\frac{\partial^2}{\partial p^2} - 2\frac{\partial}{\partial p}\frac{\partial}{\partial q} + \frac{\partial^2}{\partial q^2} \right).$$

When the operators are substituted into the wave equation, it is simplified as follows:

$$\frac{\partial^2 \mathbf{U}}{\partial \zeta^2} - \frac{1}{v^2}\frac{\partial^2 \mathbf{U}}{\partial t^2} =$$
$$= \frac{\partial^2 \mathbf{U}}{\partial p^2} + 2\frac{\partial^2 \mathbf{U}}{\partial p \partial q} + \frac{\partial^2 \mathbf{U}}{\partial q^2} - \frac{1}{v^2}v^2\left(\frac{\partial^2 \mathbf{U}}{\partial p^2} - 2\frac{\partial^2 \mathbf{U}}{\partial p \partial q} + \frac{\partial^2 \mathbf{U}}{\partial q^2} \right) =$$
$$= 4\frac{\partial^2 \mathbf{U}}{\partial p \partial q} = 0 \Rightarrow \boxed{\frac{\partial^2 \mathbf{U}}{\partial p \partial q} = 0}.$$

The general solution of the transformed wave equation is the function

$$\mathbf{U} = \mathbf{U}_1(p) + \mathbf{U}_2(q) = \mathbf{U}_1\left((\mathbf{r}, \mathbf{s}) - vt \right) + \mathbf{U}_2\left((\mathbf{r}, \mathbf{s}) + vt \right).$$

Another approach to the solution uses separation of variables. We will look for the solution in complex form

$$\mathbf{U}(\mathbf{r}, t) = \mathbf{U}_0(\mathbf{r})e^{-i\omega t}.$$

When substituting into the wave equation, we obtain:

$$\frac{\partial^2 \mathbf{U}}{\partial t^2} = -\omega^2 e^{-i\omega t}\mathbf{U}_0(\mathbf{r}), \quad \nabla^2 \mathbf{U} = e^{-i\omega t}\nabla^2 \mathbf{U}_0(\mathbf{r}),$$

$$\nabla^2 \mathbf{U} - \frac{1}{v^2}\frac{\partial^2 \mathbf{U}}{\partial t^2} = 0 \Longrightarrow \nabla^2 \mathbf{U}_0 + \frac{\omega^2}{v^2}\mathbf{U}_0 = 0.$$

Let's introduce some scalar quantities: *wave number* $k = \omega/v$, $k_0 = \omega/c$, *wave vector* $\mathbf{k} = k\mathbf{s}$. Let us recall that c — the speed of light in a vacuum, v — the speed of an electromagnetic wave in a medium, $n = \sqrt{\varepsilon\mu}$ — the refractive index of the medium, $\mathbf{s}$ — direction of wave propagation. The velocities v

and c are related by the relations $v = c/\sqrt{\varepsilon\mu} = c/n$, so the wave number can also be written as $k = \omega/v = \omega\sqrt{\varepsilon\mu}/c = k_0 n$. Now the equation for $\mathbf{U}_0$ can be rewritten as:

$$\left(\nabla^2 + k^2\right)\mathbf{U}_0 = 0.$$

This equation is called *Helmholtz equation* (homogeneous Helmholtz equation). In the general case, its solution can be expressed in special functions, but in the case of a plane wave, the general solution can be written in the following form:

$$\mathbf{U}_0(\mathbf{r}) = \mathbf{u}_0(\mathbf{r})e^{ik(\mathbf{s},\mathbf{r})} = \mathbf{u}_0(\mathbf{r})e^{ik_0 n(\mathbf{s},\mathbf{r})}.$$

4. Derivation of the eikonal equation

We will also consider a strictly monochromatic harmonic wave, the intensity vectors of which can be written in the following form:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r})e^{-i\omega t},$$
$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0(\mathbf{r})e^{-i\omega t},$$

where $\mathbf{r} = (x, y, z)^T$ — radius vector of a point in space in a Cartesian coordinate system, ω — cyclic frequency. We also introduce the quantity $k_0 = \omega/c = 2\pi/\lambda_0$, where λ_0 is the wavelength in vacuum.

Let's substitute expressions for a monochromatic wave into Maxwell's equations. We sequentially calculate all differential operators:

$$\boldsymbol{\nabla} \times \mathbf{H} = \boldsymbol{\nabla} \times \left(\mathbf{H}_0 e^{-i\omega t}\right) = e^{-i\omega t}\boldsymbol{\nabla} \times \mathbf{H}_0,$$
$$\boldsymbol{\nabla} \times \mathbf{E} = \boldsymbol{\nabla} \times \left(\mathbf{E}_0 e^{-i\omega t}\right) = e^{-i\omega t}\boldsymbol{\nabla} \times \mathbf{E}_0.$$

Using the material equations $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ we replace $\mathbf{D}$ and $\mathbf{B}$ everywhere through $\mathbf{E}$ and $\mathbf{H}$, taking into account that $\varepsilon(\mathbf{r}) = \varepsilon(x, y, z)$ and $\mu(\mathbf{r}) = \mu(x, y, z)$:

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \boldsymbol{\nabla} \cdot (\varepsilon(x, y, z)\mathbf{E}) = e^{-i\omega t}\boldsymbol{\nabla} \cdot (\varepsilon\mathbf{E}_0),$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = \boldsymbol{\nabla} \cdot (\mu(x, y, z)\mathbf{H}) = e^{-i\omega t}\boldsymbol{\nabla} \cdot (\mu\mathbf{H}_0).$$

Let us replace $\mathbf{D}$ and $\mathbf{B}$ also in the expressions for derivatives, taking into account that ε and μ do not depend on time, as well as $\mathbf{E}_0$ with $\mathbf{H}_0$ from the formulas $\mathbf{E}(x, y, z, t) = \mathbf{E}_0(x, y, z)e^{-i\omega t}$, $\mathbf{H}(x, y, z, t) = \mathbf{H}_0(x, y, z)e^{-i\omega t}$:

$$\frac{\partial\mathbf{D}}{\partial t} = \frac{\partial}{\partial t}\left(\varepsilon\mathbf{E}_0 e^{-i\omega t}\right) = \varepsilon(x, y, z)\mathbf{E}_0(x, y, z)\frac{\partial e^{-i\omega t}}{\partial t} = -i\varepsilon\omega\mathbf{E}_0 e^{-i\omega t},$$
$$\frac{\partial\mathbf{B}}{\partial t} = \frac{\partial}{\partial t}\left(\mu\mathbf{H}_0 e^{-i\omega t}\right) = \mu(x, y, z)\mathbf{H}_0(x, y, z)\frac{\partial e^{-i\omega t}}{\partial t} = -i\mu\omega\mathbf{H}_0 e^{-i\omega t}.$$

Let's substitute the resulting expressions into the equation (5):

$$\boldsymbol{\nabla}\times\mathbf{H} - \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} = 0 \Rightarrow \cancel{e^{-i\omega t}}\boldsymbol{\nabla}\times\mathbf{H}_0 + i\varepsilon\frac{\omega}{c}\mathbf{E}_0\cancel{e^{-i\omega t}} = 0 \Rightarrow \boxed{\boldsymbol{\nabla} \times \mathbf{H}_0 + i\varepsilon k_0\mathbf{E}_0 = 0},$$

then into the equation (6):

$$\boldsymbol{\nabla}\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0\Rightarrow \cancel{e^{-i\omega t}}\boldsymbol{\nabla}\times\mathbf{E}_0-i\varepsilon\frac{\omega}{c}\mathbf{H}_0\cancel{e^{-i\omega t}}=0\Rightarrow\boxed{\boldsymbol{\nabla}\times\mathbf{E}_0-i\varepsilon k_0\mathbf{H}_0=0},$$

into the equation (7):

$$\boldsymbol{\nabla}\cdot\mathbf{D}=0\Rightarrow e^{-i\omega t}\boldsymbol{\nabla}\cdot(\varepsilon\mathbf{E}_0)=0\Rightarrow\boxed{\boldsymbol{\nabla}\cdot(\varepsilon\mathbf{E}_0)=0},$$

and finally into the equation (8):

$$\boldsymbol{\nabla}\cdot\mathbf{B}=0\Rightarrow e^{-i\omega t}\boldsymbol{\nabla}\cdot(\mu\mathbf{H}_0)=0\Rightarrow\boxed{\boldsymbol{\nabla}\cdot(\mu\mathbf{H}_0)=0}.$$

As a result, the system of equations (5)–(8) takes the following simplified form:

$$\begin{cases}\boldsymbol{\nabla}\times\mathbf{H}_0+i\varepsilon k_0\mathbf{E}_0=0,\\ \boldsymbol{\nabla}\times\mathbf{E}_0-i\mu k_0\mathbf{H}_0=0,\\ \boldsymbol{\nabla}\cdot(\varepsilon\mathbf{E}_0)=0,\\ \boldsymbol{\nabla}\cdot(\mu\mathbf{H}_0)=0.\end{cases}\tag{9}$$

Let us make another simplification by assuming that

$$\mathbf{E}_0(x,y,z)=\mathbf{e}(x,y,z)\exp\left(ik_0u(x,y,z)\right)=\mathbf{e}(\mathbf{r})\exp\left(ik_0u(\mathbf{r})\right),$$
$$\mathbf{H}_0(x,y,z)=\mathbf{h}(x,y,z)\exp\left(ik_0u(x,y,z)\right)=\mathbf{h}(\mathbf{r})\exp\left(ik_0u(\mathbf{r})\right),$$

where $u(x,y,z)=u(\mathbf{r})$ is a scalar real function called *optical path*, and $\mathbf{e}$ and $\mathbf{h}$ — vector position functions. Let's calculate the differential operators again, this time from $\mathbf{E}_0$ and $\mathbf{H}_0$, using the formulas (18):

$$\boldsymbol{\nabla}\times\mathbf{H}_0=\boldsymbol{\nabla}\times\left(e^{ik_0u(\mathbf{r})}\mathbf{h}(\mathbf{r})\right)=e^{ik_0u(\mathbf{r})}\boldsymbol{\nabla}\times\mathbf{h}+\boldsymbol{\nabla}\left(e^{ik_0u(\mathbf{r})}\right)\times\mathbf{h}.$$

The gradient of the function $e^{ik_0u(\mathbf{r})}$ is calculated as follows:

$$\boldsymbol{\nabla}\left(e^{ik_0u(\mathbf{r})}\right)=\left(\frac{\partial e^{ik_0u(\mathbf{r})}}{\partial x},\frac{\partial e^{ik_0u(\mathbf{r})}}{\partial y},\frac{\partial e^{ik_0u(\mathbf{r})}}{\partial z}\right)=$$
$$=ik_0e^{ik_0u(\mathbf{r})}\left(\frac{\partial u(x,y,z)}{\partial x},\frac{\partial u(x,y,z)}{\partial y},\frac{\partial u(x,y,z)}{\partial z}\right)=$$
$$=ik_0e^{ik_0u(\mathbf{r})}\boldsymbol{\nabla}u(x,y,z).$$

As a result, the term $\boldsymbol{\nabla}\times\mathbf{H}_0$ of the first equation of the system (9) takes the form:

$$\boldsymbol{\nabla}\times\mathbf{H}_0=(\boldsymbol{\nabla}\times\mathbf{h}+ik_0\boldsymbol{\nabla}u\times\mathbf{h})e^{ik_0u(\mathbf{r})}.\tag{10}$$

In a completely similar way, we obtain the expression for $\boldsymbol{\nabla}\times\mathbf{E}_0$ in the second equation of the system (9):

$$\boldsymbol{\nabla}\times\mathbf{E}_0=(\boldsymbol{\nabla}\times\mathbf{e}+ik_0\boldsymbol{\nabla}u\times\mathbf{e})e^{ik_0u(\mathbf{r})}.\tag{11}$$

The computation of divergence is somewhat more complicated because the formula (18) will have to be applied twice. The first time we use it to write down the expression $\boldsymbol{\nabla}\cdot\varepsilon\mathbf{E}_0$:

$$\boldsymbol{\nabla}\cdot\varepsilon\mathbf{E}_0 = \varepsilon(\mathbf{r})\boldsymbol{\nabla}\cdot\mathbf{E}_0 + \left(\boldsymbol{\nabla}\varepsilon, \mathbf{E}_0\right).$$

Next, we use it to calculate $\boldsymbol{\nabla}\cdot\mathbf{E}_0$, where instead of $\mathbf{E}_0$ we substitute the expression $\mathbf{E}_0 = \mathbf{e}(\mathbf{r})\exp\left(ik_0u(\mathbf{r})\right)$:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{E}_0 = \boldsymbol{\nabla}\cdot\left[\mathbf{e}(\mathbf{r})e^{ik_0u(\mathbf{r})}\right] = e^{ik_0u(\mathbf{r})}\boldsymbol{\nabla}\cdot\mathbf{e} + \left(\boldsymbol{\nabla}\left(e^{ik_0u(\mathbf{r})}\right), \mathbf{e}\right) = \\ = e^{ik_0u(\mathbf{r})}\boldsymbol{\nabla}\cdot\mathbf{e} + ik_0e^{ik_0u(\mathbf{r})}\left(\boldsymbol{\nabla}u, \mathbf{e}\right) = \left(\boldsymbol{\nabla}\cdot\mathbf{e} + ik_0(\boldsymbol{\nabla}u, \mathbf{e})\right)e^{ik_0u(\mathbf{r})},\end{aligned}$$

$$(\boldsymbol{\nabla}\varepsilon, \mathbf{E}_0) = (\boldsymbol{\nabla}\varepsilon, \mathbf{e})e^{ik_0u(\mathbf{r})}.$$

As a result, the third equation of the system (9) takes the form:

$$\boldsymbol{\nabla}\cdot(\varepsilon(\mathbf{r})\mathbf{E}_0(\mathbf{r})) = [\varepsilon(\mathbf{r})\boldsymbol{\nabla}\cdot\mathbf{e}(\mathbf{r}) + ik_0\varepsilon(\mathbf{r})(\boldsymbol{\nabla}u(\mathbf{r}), \mathbf{e}(\mathbf{r})) + (\boldsymbol{\nabla}\varepsilon(\mathbf{r}), \mathbf{e}(\mathbf{r}))]e^{ik_0u(\mathbf{r})}.$$

In a completely similar way, we obtain an expression for the magnetic field strength, that is, the fourth equation of the system (9):

$$\boldsymbol{\nabla}\cdot(\mu(\mathbf{r})\mathbf{H}_0(\mathbf{r})) = [\mu(\mathbf{r})\boldsymbol{\nabla}\cdot\mathbf{h} + (\boldsymbol{\nabla}\mu(\mathbf{r}), \mathbf{h}) + ik_0\mu(\mathbf{r})(\boldsymbol{\nabla}u(\mathbf{r}), \mathbf{h})]e^{ik_0u(\mathbf{r})}.$$

After substitution into Maxwell's equations, we obtain:

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{H}_0 + i\varepsilon k_0\mathbf{E}_0 = \mathbf{0} \Rightarrow \underbrace{\boldsymbol{\nabla}\times\mathbf{h} + ik_0\boldsymbol{\nabla}u\times\mathbf{h}}_{(10)} + i\varepsilon k_0\mathbf{e} = \mathbf{0} \Rightarrow \\ \Rightarrow \boldsymbol{\nabla}u\times\mathbf{h} + \varepsilon\mathbf{e} = -\frac{1}{ik_0}\boldsymbol{\nabla}\times\mathbf{h},\end{aligned}$$

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{E}_0 - i\mu k_0\mathbf{H}_0 = \mathbf{0} \Rightarrow \underbrace{\boldsymbol{\nabla}\times\mathbf{e} + ik_0\boldsymbol{\nabla}u\times\mathbf{e}}_{(11)} - i\mu k_0\mathbf{h} = \mathbf{0} \Rightarrow \\ \Rightarrow \boldsymbol{\nabla}u\times\mathbf{e} - \mu\mathbf{h} = -\frac{1}{ik_0}\boldsymbol{\nabla}\times\mathbf{e},\end{aligned}$$

$$\boldsymbol{\nabla}\cdot(\varepsilon\mathbf{E}_0) = 0 \Rightarrow \varepsilon\boldsymbol{\nabla}\cdot\mathbf{e} + ik_0\varepsilon(\boldsymbol{\nabla}u, \mathbf{e}) + (\boldsymbol{\nabla}\varepsilon, \quad \mathbf{e}) = 0,$$

$$ik_0\varepsilon(\boldsymbol{\nabla}u, \mathbf{e}) = -(\boldsymbol{\nabla}\varepsilon, \mathbf{e}) - \varepsilon\boldsymbol{\nabla}\cdot\mathbf{e} = 0 \Rightarrow (\boldsymbol{\nabla}u, \mathbf{e}) = -\frac{1}{ik_0}\left[\left(\frac{1}{\varepsilon}\boldsymbol{\nabla}\varepsilon, \mathbf{e}\right) + \boldsymbol{\nabla}\cdot\mathbf{e}\right].$$

Since

$$\boldsymbol{\nabla}(\ln\varepsilon) = \left(\frac{\partial\ln\varepsilon}{\partial x}, \frac{\partial\ln\varepsilon}{\partial y}, \frac{\partial\ln\varepsilon}{\partial z}\right) = \frac{1}{\varepsilon}\left(\frac{\partial\varepsilon}{\partial x}, \frac{\partial\varepsilon}{\partial y}, \frac{\partial\varepsilon}{\partial z}\right) = \frac{1}{\varepsilon}\boldsymbol{\nabla}\varepsilon,$$

$$(\boldsymbol{\nabla}u, \mathbf{e}) = -\frac{1}{ik_0}((\boldsymbol{\nabla}(\ln\varepsilon), \mathbf{e}) + \boldsymbol{\nabla}\cdot\mathbf{e}).$$

Calculations for the magnetic field are carried out in a completely similar way, resulting in the fourth equation:

$$(\boldsymbol{\nabla} u, \mathbf{h}) = -\frac{1}{ik_0}((\boldsymbol{\nabla}(\ln\mu), \mathbf{h}) + \boldsymbol{\nabla}\cdot\mathbf{h}),$$

$$\begin{cases} \boldsymbol{\nabla} u \times \mathbf{h} + \varepsilon\mathbf{e} = -\dfrac{1}{ik_0}\boldsymbol{\nabla}\times\mathbf{h}, \\ \boldsymbol{\nabla} u \times \mathbf{e} - \mu\mathbf{h} = -\dfrac{1}{ik_0}\boldsymbol{\nabla}\times\mathbf{e}, \\ (\boldsymbol{\nabla} u, \mathbf{e}) = -\dfrac{1}{ik_0}\left((\boldsymbol{\nabla}\left(\ln\varepsilon\right), \mathbf{e}) + \boldsymbol{\nabla}\cdot\mathbf{e}\right), \\ (\boldsymbol{\nabla} u, \mathbf{h}) = -\dfrac{1}{ik_0}\left((\boldsymbol{\nabla}\left(\ln\mu\right), \mathbf{h}) + \boldsymbol{\nabla}\cdot\mathbf{h}\right). \end{cases} \tag{12}$$

The third and fourth equations from this system follow from the first two. This can be proven by scalarly multiplying the first two equations by $\boldsymbol{\nabla} u$ and using the fact that the result of a vector product is orthogonal to both of its factors:

$$\underbrace{(\boldsymbol{\nabla} u, \boldsymbol{\nabla} u\times\mathbf{h})}_{=0} + \varepsilon(\boldsymbol{\nabla} u, \mathbf{e}) = 0 \Rightarrow (\boldsymbol{\nabla} u, \mathbf{e}) = 0.$$

We consider only the first two equations. Let's express $\mathbf{h}$ from the second equation through u and $\mathbf{e}$ and substitute it into the first:

$$\mathbf{h} = \frac{1}{\mu}\boldsymbol{\nabla} u\times\mathbf{e} \Rightarrow \boldsymbol{\nabla} u\times\left(\frac{1}{\mu}\boldsymbol{\nabla} u\times\mathbf{e}\right) + \varepsilon\mathbf{e} = \mathbf{0} \Rightarrow \boldsymbol{\nabla} u\times\boldsymbol{\nabla} u\times\mathbf{e} + \varepsilon\mu\mathbf{e} = 0.$$

For the vector product the following identity holds: $\mathbf{a}\times\mathbf{b}\times\mathbf{c} = \mathbf{b}(\mathbf{a},\mathbf{c}) - \mathbf{c}(\mathbf{a},\mathbf{b})$ from which it follows

$$\boldsymbol{\nabla} u\times\boldsymbol{\nabla} u\times\mathbf{e} = \boldsymbol{\nabla} u(\boldsymbol{\nabla} u,\mathbf{e}) - \mathbf{e}(\boldsymbol{\nabla} u,\boldsymbol{\nabla} u) = \boldsymbol{\nabla} u(\boldsymbol{\nabla} u,\mathbf{e}) - \mathbf{e}\|\boldsymbol{\nabla} u\|^2,$$

$$\boldsymbol{\nabla} u(\boldsymbol{\nabla} u,\mathbf{e}) - \mathbf{e}\|\boldsymbol{\nabla} u\|^2 + \varepsilon\mu\mathbf{e} = 0.$$

From the third equation of the system (12) it follows that $(\boldsymbol{\nabla} u, \mathbf{e})$, therefore

$$-\mathbf{e}\|\boldsymbol{\nabla} u\|^2 + \varepsilon\mu\mathbf{e} = 0 \Rightarrow \mathbf{e}\|\boldsymbol{\nabla} u\|^2 = \varepsilon\mu\mathbf{e}.$$

Equating the coefficients in front of the vector $\mathbf{e}$ and taking into account that $n(\mathbf{r}) = \sqrt{\varepsilon(\mathbf{r})\mu(\mathbf{r})}$ we write the equation:

$$\|\boldsymbol{\nabla} u\|^2 = n^2(\mathbf{r}), \tag{13}$$

which is the *eikonal equation*. The function $u(\mathbf{r}) = u(x,y,z)$ is also called *eikonal*, and the surfaces $u(x,y,z) = \mathrm{const}$ — *geometric wave fronts*.

In component form in Cartesian coordinates equation (13) becomes:

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \varepsilon(x,y,z)\mu(x,y,z) = n^2(x,y,z),$$

$$\|\boldsymbol{\nabla} u\|^2 = (\boldsymbol{\nabla} u, \boldsymbol{\nabla} u) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2.$$

5. Derivation of eikonal in covariant form

Let us demonstrate the derivation of the eikonal equation using the tensor formalism.

5.1. Vector operators in covariant form

Vector operators in covariant form:
— $\nabla_{\vec{V}}$ is covariant derivative with respect to the vector field $\vec{v}$;
— $\vec{e}_i = \dfrac{\partial}{\partial x^i}$ is coordinate basis, $\nabla\ \vec{e}_i = \nabla_j$;
— $\varepsilon_{ijk} = \varepsilon^{ijk}$ is Levi–Civita symbol;
— $e_{ijk} = \sqrt{|g|}\varepsilon_{ijk}$, $e^{ijk} = \dfrac{1}{\sqrt{|g|}}\varepsilon^{ijk}$ are alternating tensors (Levi–Civita tensors);
— $\boldsymbol{\nabla} f = \nabla_i f = \partial_i f$, f is scalar field;
— $\boldsymbol{\nabla} \cdot f = \nabla_i V^i = \dfrac{1}{\sqrt{g}}\partial_i(\sqrt{g}V^i)$;
— $\vec{x} = (x^1, x^2, x^3)^T$ is contravariant vector;
— $\boldsymbol{\nabla} \times \vec{V} = e^{ijk}\nabla_j V_k = e^{ijk}\partial_j V_k = \dfrac{1}{\sqrt{g}}\varepsilon^{ijk}\partial_j V_k$.

5.2. Maxwell's equations without currents and charges

The strength of the electric and magnetic fields in the form of a covector (denoted by "$\sim$" above the letter, the designations can be changed), and D and B are vectors:

$$\tilde{E} = (E_1, E_2, E_3), \quad \tilde{H} = (H_1, H_2, H_3), \quad \vec{D} = (D^1, D^2, D^3)^T, \quad \vec{B} = (B^1, B^2, B^3)^T.$$

Material equations: $B^i = \mu^{ij} H_j$, $D^i = \varepsilon^{ij} E_j$.
Vector, covector fields: $\tilde{E}(\vec{x}, t)$, $\tilde{H}(\vec{x}, t)$, $\vec{D}(\tilde{x}, t)$, $\vec{B}(\tilde{x}, t)$.
Tensor fields: $\mu^{ij}(\vec{x})$, $\varepsilon^{ij}(\vec{x})$.

5.3. Vector-differential form of writing Maxwell's equations

$$\begin{cases} \boldsymbol{\nabla} \times \vec{H} - \dfrac{1}{c}\dfrac{\partial \vec{D}}{\partial t} = 0, \\ \boldsymbol{\nabla} \times \vec{E} + \dfrac{1}{c}\dfrac{\partial \vec{B}}{\partial t} = 0, \\ \boldsymbol{\nabla} \cdot \vec{D} = 0, \\ \boldsymbol{\nabla} \cdot \vec{B} = 0; \end{cases}$$

$$\begin{cases} \dfrac{1}{\sqrt{g}}\varepsilon^{ijk}\partial_j E_k + \dfrac{1}{c}\dfrac{\mathrm{d}B^i}{\mathrm{d}t} = 0, \\ \dfrac{1}{\sqrt{g}}\varepsilon^{ijk}\mathrm{d}_j H_k - \dfrac{1}{c}\dfrac{\mathrm{d}D^i}{\mathrm{d}t} = 0, \\ \dfrac{1}{\sqrt{g}}\partial_i(\sqrt{g}D^i) = 0, \\ \dfrac{1}{\sqrt{g}}\partial_i(\sqrt{g}B^i) = 0. \end{cases}$$

5.4. Monochromatic harmonic wave

Assumption No. 1: Monochromatic harmonic wave:

$$\begin{cases} E_k = E_{0k}e^{-i\omega t},\ H_k = H_{0k}e^{-i\omega t},\ D^k = \varepsilon^{kl}E_l = \varepsilon^{kl}E_{0l}e^{-i\omega t}, \\ B^k = \mu^{kl}H_l = \mu^{kl}H_{0k}e^{-i\omega t}, \\ \dfrac{\mathrm{d}D^i}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon^{ij}E_{0j}e^{-i\omega t}\right) = -i\omega\varepsilon^{ij}E_{0j}, \\ \dfrac{\mathrm{d}B^i}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\mu^{ij}H_{0j}e^{-i\omega t}\right) = -i\omega\mu^{ij}H_{0j}, \\ \partial_i(\sqrt{g}D^i) = \partial_i(\sqrt{g}\varepsilon^{ij}E_{0j}e^{-i\omega t}) = e^{-i\omega t}\partial_i(\sqrt{g}\varepsilon^{ij}E_{0j}), \\ \partial_i(\sqrt{g}B^i) = \partial_i(\sqrt{g}\mu^{ij}H_{0j}e^{-i\omega t}) = e^{-i\omega t}\partial_i(\sqrt{g}\mu^{ij}H_{0j}). \end{cases}$$

Formulas:

$$\frac{1}{\sqrt{g}}\varepsilon^{ijk}\partial_j E_{0k} - ik_0\mu^{ij}H_{0j} = 0, \tag{14}$$

$$\frac{1}{\sqrt{g}}\varepsilon^{ijk}\partial_j H_{0k} + ik_0\varepsilon^{ij}E_{0j} = 0, \tag{15}$$

$$\partial_i(\sqrt{g}\varepsilon^{ij}E_{0j}) = 0, \tag{16}$$

$$\partial_i(\sqrt{g}\mu^{ij}H_{0j}) = 0. \tag{17}$$

Assumption No. 2: $E_{0k} = e_k e^{ik_0 u(\vec{x})}$, $H_{0k} = h_k e^{ik_0 u(\vec{x})}$, where $u(\vec{x})$ is an eikonal:

$$\partial_j E_{0k} = (\partial_j e_k) e^{ik_0 u} + e_k e^{ik_0 u} ik_0 \partial_j u = (\partial_j e_k + ik_0 e_k \partial_j u) e^{ik_0 u},$$
$$\partial_j H_{0k} = (\partial_j h_k) e^{ik_0 u} + h_k e^{ik_0 u} ik_0 \partial_j u = (\partial_j h_k + ik_0 h_k \partial_j u) e^{ik_0 u}.$$

From the equation (16):

$$\partial_i(\sqrt{g}\varepsilon^{ij} e_j e^{ik_0 u}) =$$
$$= \frac{\partial\sqrt{g}}{\partial x^i}\varepsilon^{ij} e_j e^{ik_0 u} + \sqrt{g}\frac{\partial \varepsilon^{ij}}{\partial x^i} e_j e^{ik_0 u} + \sqrt{g}\varepsilon^{ij}\frac{\partial e_j}{\partial x^i} e^{ik_0 u} + \sqrt{g}\varepsilon^{ij} e_j ik_0 e^{ik_0 u}\frac{\partial u}{\partial x^i} =$$
$$= (\partial_i\sqrt{g}\varepsilon^{ij} e_j + \sqrt{g}\partial_i\varepsilon^{ij} e_j + \sqrt{g}\varepsilon^{ij}\partial_i e_j + ik_0\sqrt{g}\varepsilon^{ij} e_j \partial_i u) e^{ik_0 u} = 0,$$

$$\partial_i\sqrt{g}\varepsilon^{ij} e_j + \sqrt{g}\partial_i\varepsilon^{ij} e_j + \sqrt{g}\varepsilon^{ij}\partial_i e_j + ik_0\sqrt{g}\varepsilon^{ij} e_j \partial_i u = 0,$$
$$\sqrt{g}\varepsilon^{ij} e_j \partial_i u = \frac{-1}{ik_0}(\partial_i\sqrt{g}\varepsilon^{ij} e_j + \sqrt{g}\partial_i\varepsilon^{ij} e_j + \sqrt{g}\varepsilon^{ij}\partial_i e_j).$$

Similarly from the equation (17):

$$\sqrt{g}\mu^{ij} h_j \partial_i u = -\frac{1}{ik_0}\left(\partial_i\sqrt{g}\mu^{ij} h_j + \sqrt{g}\partial_i\mu^{ij} + \sqrt{g}\mu^{ij}\partial_i h_j\right).$$

Provided that λ is small, ω is large, $\Rightarrow k_0$ is large, and $\dfrac{1}{k_0}$ is small $\Rightarrow$ we obtain:

$$\begin{cases} \sqrt{g}\varepsilon^{ij} e_j \partial_i u = 0, \\ \sqrt{g}\mu^{ij} h_j \partial_i u = 0; \end{cases} \qquad \begin{cases} \varepsilon^{ij} e_j \partial_i u = 0, \\ \mu^{ij} h_j \partial_i u = 0 \end{cases}$$

to transform:

$$\frac{1}{\sqrt{g}}\varepsilon^{ijk}(\partial_j e_k + ik_0 e_k \partial_j u) - ik_0\mu^{ij} h_j = 0 \Rightarrow -\frac{1}{\sqrt{g}}\varepsilon^{ijk} e_k \partial_j u + \mu^{ij} h_0 = \frac{1}{ik_0}\frac{1}{\sqrt{g}}\varepsilon^{ijk}\partial_j e_k,$$

$$\frac{1}{\sqrt{g}}\varepsilon^{ijk}\partial_j h_k + \frac{1}{\sqrt{g}}\varepsilon^{ijk} ik_0 h_k \partial_j u + ik_0\varepsilon^{ij} e_j = 0,$$
$$\frac{1}{\sqrt{g}}\varepsilon^{ijk} h_k \partial_j u + \varepsilon^{ij} e_j = -\frac{1}{ik_0}\frac{1}{\sqrt{g}}\varepsilon^{ijk}\partial_j h_k.$$

Maxwell's equations are reduced to the following form:

$$\begin{cases} \varepsilon^{ijk} e_k \partial_j u - \sqrt{g}\mu^{ij} h_j = 0, \\ \varepsilon^{ijk} h_k \partial_j u + \sqrt{g}\varepsilon^{ij} e_j = 0, \\ \varepsilon^{ij} e_j \partial_i u = 0, \\ \mu^{ij} h_j \partial_i u = 0, \end{cases}$$

where ε^{ijk} is Levi-Civita symbol, ε^{ij} is permittivity, subject to $k_0 \to \infty$.

From the first equation we express h_j and substitute it into the second:

$$\mu_{li}^{-1}\varepsilon^{ijk}e_k\partial_j u - \sqrt{g}\mu_{li}^{-1}\mu^{ij}h_j = 0.$$

Let's make the replacement: $\mu_{li}^{-1}\mu^{ij} = g_l^j$:

$$\mu_{li}^{-1}\varepsilon^{ijk}e_k\partial_j u - \sqrt{g}g_l^j h_j = 0 \Rightarrow \sqrt{g}h_l = \mu_{li}^{-1}\varepsilon^{ijk}e_k\partial_j u \Rightarrow h_l = \frac{1}{\sqrt{g}}\mu_{li}^{-1}\varepsilon^{ijk}e_k\partial_j u.$$

We transform the indices to substitute into the second equation:

$$h_k = \frac{1}{\sqrt{g}}\mu_{kl}^{-1}\varepsilon^{lmn}e_n\partial_m u,$$
$$\varepsilon^{ijk}\frac{1}{\sqrt{g}}\mu_{kl}^{-1}\varepsilon^{lmn}e_n\partial_m u\partial_j u + \sqrt{g}\varepsilon^{ij}e_j = 0,$$
$$\varepsilon^{ijk}\mu_{kl}^{-1}\varepsilon^{lmn}e_n\partial_m u\partial_j u + g\varepsilon^{ij}e_j = 0,$$
$$\varepsilon^{ijk}\mu_{kl}^{-1}\varepsilon^{lmn}\partial_m u\partial_j u e_n + g\varepsilon^{in}e_n = 0.$$

The eikonal equation (13) takes the form:

$$g^{ij}\partial_i u\partial_j u = \varepsilon^{ij}\mu_{ij}.$$

6. Conclusion

We hope that our work clarifies the process of derivation of the eikonal equation. And allows us to better understand the hierarchy of models in electrodynamics in general, and in optics in particular. The questions of solving the eikonal equation [7, 8] we left outside the boundaries of our consideration.

Appendix. Vector analysis

If at each point P of a certain spatial region of the Euclidean space $\mathbb{R}^n$ some scalar or vector quantity is associated, then they say that a *field* (*scalar* or *vector*).

— Examples of vector fields include the velocity field $\mathbf{v}(x, y, z)$, the force field $\mathbf{F}(x, y, z)$, the electrical intensity field $\mathbf{E}(x, y, z)$.

— Examples of scalar fields: temperature field $T(x, y, z)$, electric potential field $\varphi(x, y, z)$.

Everywhere below we consider a three-dimensional point Euclidean space on which a Cartesian coordinate system is introduced. We denote the vectors (basis vectors) of this coordinate system as $\langle \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z \rangle$. The coordinates of a point are specified by the radius vector $\mathbf{r} = (x, y, z)^T$, which is plotted from the origin O. Along with notation of coordinates x, y, z, it is sometimes convenient to use indices: x^1, x^2, x^3, and also write the radius vector in the form $\mathbf{x} = (x^1, x^2, x^3)^T$. Index notation makes it possible to briefly write formulas using the summation sign Σ, which is especially convenient if a non-unit metric is used.

A scalar field in some region of space $\mathbb{R}^3$ is a real-valued function f:

$$f\colon \mathbb{R}^3 \to \mathbb{R}, \quad f(x,y,z) = f(\mathbf{r}) \in \mathbb{R}.$$

In turn, a vector field in a region of space $\mathbb{R}^3$ is a vector-valued function $\mathbf{V}$:

$$\mathbf{V}\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad \mathbf{V}(x,y,z) = \mathbf{V}(\mathbf{r}) = V_x(\mathbf{r})\mathbf{e}_x + V_y(\mathbf{r})\mathbf{e}_y + V_z(\mathbf{r})\mathbf{e}_z \in \mathbb{R}^3.$$

The *Gradient* of the scalar field $f(\mathbf{r})$ is a vector calculated in Cartesian coordinates as follows:

$$\boldsymbol{\nabla} f(x,y,z) = \mathrm{grad} f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right), \quad \boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

The sign nabla ∇ denotes the Hamiltonian vector differential operator. In order to emphasize its "vectority", the symbol $\boldsymbol{\nabla}$ is written in bold.

To simplify the presentation, we made some inaccuracies in the presentation, which should be mentioned separately.

— Strictly speaking, the gradient is a covector. Our definition reflects this by writing the vector components in a row rather than a column.

— The definition of the gradient is based on the Cartesian coordinate system. A more general definition should be given in a componentless form.

The scalar field $f(\mathbf{r})$ generates a vector field $\boldsymbol{\nabla} f$, which characterizes the direction of the greatest change in the scalar field $f(\mathbf{r})$.

Divergence of the vector field $\mathbf{V} = (V_x, V_y, V_z)^T$ is a scalar, calculated in Cartesian coordinates as follows:

$$\boldsymbol{\nabla} \cdot \mathbf{V} = \mathrm{div}\mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \sum_{i=1}^{3} \frac{\partial V^i}{\partial x^i}.$$

Here "$\cdot$" denotes the scalar multiplication operation $\boldsymbol{\nabla} \cdot \mathbf{V} = (\boldsymbol{\nabla}, \mathbf{V})$.

The *Rotor* of a vector field $\mathbf{V}$ is a vector calculated in Cartesian coordinates as follows:

$$\boldsymbol{\nabla} \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix} =$$
$$= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\mathbf{e}_z.$$

Highlight also that the rotor is not a vector in the strict sense. In classical vector analysis it is called a *pseudovector*, but a deeper geometric meaning is revealed only when tensor algebra is involved, where the rotor can be represented either as a 2-form or as a bivector.

Also, when writing the wave equation, the Laplace operator will be used, which is written in the following form:

$$\nabla^2 = (\nabla, \nabla) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

We will also need the following two relations [5]:

$$\begin{gathered} \boldsymbol{\nabla} \times f\mathbf{V} = f\,\boldsymbol{\nabla} \times \mathbf{V} + \boldsymbol{\nabla} f \times \mathbf{V}, \\ \boldsymbol{\nabla} \cdot f\mathbf{V} = f\,\boldsymbol{\nabla} \cdot \mathbf{V} + (\boldsymbol{\nabla} f, \mathbf{V}). \end{gathered} \tag{18}$$

A vector field is called *potential* if there exists a scalar field $f(x, y, z)$ such that

$$\begin{gathered} \mathbf{V} = \boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right), \\ \mathrm{d}f = V_x\,\mathrm{d}x + V_y\,\mathrm{d}y + V_z\,\mathrm{d}z\,. \end{gathered}$$

In turn, a vector field is called *solenoidal* (tubular) if there exists a vector field $\mathbf{U}$ such that

$$\mathbf{V} = \boldsymbol{\nabla} \times \mathbf{U}.$$

Acknowledgments

This publication has been supported by the RUDN University Scientific Projects Grant System, project No 021934-0-000 (Korolkova A. V., Gevorkyan M. N.). This paper has been supported by the RUDN University Strategic Academic Leadership Program.

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For citation:

A. V. Fedorov, C. A. Stepa, A. V. Korolkova, M. N. Gevorkyan, D. S. Kulyabov, Methodological derivation of the eikonal equation, Discrete and Continuous Models and Applied Computational Science 31 (4) (2023) 399–418. DOI: 10.22363/2658-4670-2023-31-4-399-418.

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УДК 537.8:514.762.37
DOI: 10.22363/2658-4670-2023-31-4-399-418
EDN: GCUXWK

Методический вывод уравнения эйконала

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Аннотация. Обычно при работе с уравнением эйконала ссылаются на его вывод в монографии Борна и Вольфа. Вывод этого уравнения выполнен достаточно небрежно. Для того чтобы разобраться в этом выводе, требуется определённое число имплицитных предположений. Для лучшего понимания приближения эйконала и для методических целей авторы решили повторить вывод уравнения эйконала, эксплицировав все возможные допущения. Методически предлагается следующий алгоритм вывода уравнения эйконала. Из уравнения Максвелла выводится волновое уравнение. При этом явно вводятся все условия, при которых это возможно сделать. Далее от волнового уравнения осуществляется переход к уравнению Гельмгольца. От уравнения Гельмгольца при приложении определённых допущений производится переход к уравнению эйконала. После разбора всех допущений и шагов реализуется собственно переход от уравнений Максвелла к уравнению эйконала. При выводе уравнения эйконала используется несколько формализмов. В качестве первого формализма используется стандартный формализм векторного анализа. Уравнения Максвелла и уравнение эйконала записывается в виде трёхмерных векторов. После этого и для уравнений Максвелла, и для уравнения эйконала используется ковариантный 4-мерный формализм. Результатом работы является методически выдержанное описание уравнения эйконала.

Ключевые слова: эйконал, уравнения Максвелла, волновое уравнение, векторное представление, тензорное представление