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Hodge–de Rham Laplacian and geometric criteria for gravitational waves

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Abstract. The curvature tensor $\hat{R}$ of a manifold is called harmonic, if it obeys the condition $\Delta^{(\mathrm{HR})}\hat{R} = 0$, where $\Delta^{(\mathrm{HR})} = DD^* + D^*D$ is the Hodge–de Rham Laplacian. It is proved that all solutions of the Einstein equations in vacuum, as well as all solutions of the Einstein–Cartan theory in vacuum have a harmonic curvature. The statement that only solutions of Einstein's equations of type N (describing gravitational radiation) are harmonic is refuted.

Key words and phrases: Hodge–de Rham Laplacian, harmonic curvature tensor, harmonic solutions in vacuum of Einstein equation and Einstein–Cartan theory equations

1. Introduction. Harmonic curvature tensor

In non-Euclidean spaces, in the formalism of differential forms, an external covariant differential D is defined, as well as an external covariant codifferential D^*, whose action on the p-form ω is determined by the rule:

$$D^*\omega = (-1)^p *^{-1} D * \omega,$$

where $*$ is the Hodge dualization operator mapping differential forms onto polyvectors and vice versa. The inverse operator $*^{-1}$ is determined by the rule: $*^{-1} = (-1)^{q(n-q)+s}*$, where n is the dimension of the manifold, s is the sign of the determinant of the metric, and q is the degree of the form on which the operator $*^{-1}$ acts. As a result, in the definition of D^* the operator $*^{-1}$ acts on the $D * \omega$ form of the degree $q = n - p + 1$. As a consequence, the action of the codifferential D^* can be represented in an equivalent form as: $D^*\omega = (-1)^{np+n+1+s} * D * \omega$, which for a 4-dimensional pseudo-Euclidean space and a form ω of even degree gives $D^*\omega = *D * \omega$ [1, 2].

A differential form is said to be *harmonic* if the action of the Hodge–de Rham Laplacian $\Delta^{(\mathrm{HR})} = DD^* + D^*D$ on it is identically equal to zero. In particular, if the curvature 2-form $\widehat{R}$ satisfies the condition $\Delta^{(\mathrm{HR})}\widehat{R} = 0$, then the corresponding curvature tensor is said to be *harmonic* [3, 4]. In Riemann space, this condition in coordinates is fulfilled if the equation $\nabla_\lambda R^\lambda_{\sigma\mu\nu} = 0$ is satisfied.

2. Hodge–de Rham Laplacian in gauge field theories

The Hodge–de Rham Laplacian arises in geometrized gauge theories. For example, a similar operator arises in the geometric interpretation of the Yang–Mills theory as a theory of fiber bundle, where the exterior differential d and co-differential δ are defined, as well as the operator $\Delta = d\delta + \delta d$, which in this case is called the Laplace-Beltrami Laplacian. This operator plays an essential role in the theory, since the Yang–Mills gauge field equations follow from the condition $\Delta = 0$. In this geometric interpretation the comparison of the potentials of the Yang–Mills gauge field and the linear connection of the fiber bundle is important [5].

In the geometric interpretation of the gravitational field in the Poincaré-gauge theory of gravity (PGTG) as the theory of an affine fibred space arising as a consequence of the localization of the Poincaré group, the Hodge–de Rham Laplacian $\Delta^{(\mathrm{HR})} = DD^* + D^*D$ also arises.

3. Hodge–de Rham Laplacian and geometric criterion of gravitational radiation

Earlier in the literature, when discussing various geometric criteria of gravitational radiation, one could come across the statement that those and only those Einstein spaces of type N describing gravitational radiation are harmonic (that is, for which the equality $\Delta^{(\mathrm{HR})}\widehat{R} = 0$ is satisfied). The Einstein space is understood as the space in which the Einstein equation in vacuum $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$ is satisfied.

Let us prove that this statement is erroneous, although it is given in the well-known monograph on gravitational waves [6].

4. On the property of harmonic solutions of the general theory of relativity and Poincaré-gauge theory of gravity

Let us find out what role the Hodge–de Rham Laplacian plays in the general theory of relativity and the Poincaré-gauge theory of gravity.

In authors' paper [7] the detailed calculation of the result of the action of the Hodge–de Rham Laplacian on the curvature 2-form $\widehat{R}$ of the Riemann space of general relativity is given.

The intermediate result is:

$$\Delta^{(\mathrm{HR})}\hat{R} = (1/2)\left(R^{\lambda}_{\sigma\mu\nu;\rho}{}^{;\rho} + R^{\lambda}_{\sigma\mu}{}^{\rho}_{[;\rho;\nu]} - R^{\lambda}_{\sigma\nu}{}^{\rho}_{[;\rho;\mu]}\right)\bar{e}_{\lambda}\otimes\bar{e}^{\sigma}\otimes\theta^{\mu}\wedge\theta^{\nu},$$

where $\bar{e}_{\lambda}$ is the coordinate basis of vectors; θ^{μ} are the basic 1-forms, $\wedge$ is the external multiplication; the symbol "semicolon" means covariant differentiation of the components of the curvature tensor.

We substitute into this equality the identity $R^{\lambda}_{\sigma\mu\nu;\rho}{}^{;\rho} = R^{\lambda}_{\sigma\mu}{}^{\rho}_{;\nu;\rho} - R^{\lambda}_{\sigma\nu}{}^{\rho}_{;\mu;\rho}$ obtained by the Bianchi identity contraction. As a result, we get:

$$\Delta^{(\mathrm{HR})}\hat{R} = \left(R_{\lambda\sigma\mu}{}^{\rho}_{;\rho}\right)_{;\nu}\theta^{\lambda}\wedge\theta^{\sigma}\otimes\theta^{\mu}\wedge\theta^{\nu}.$$

Based on this result, the statement was proved in [7]:

All solutions of Einstein's equation $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$ in vacuum are harmonic.

The proof is based on the equality $R_{\lambda\sigma\mu}{}^{\rho}_{;\rho} = 2R_{\mu[\lambda;\sigma]} = 0$, which is a consequence of the Bianchi identity and the Einstein equation in vacuum. The right side of this equality is called the Codazzi equation. Note that this assertion was stated in [8] (without calculations) and is also known to geometers [4].

The authors have calculated the result of the action of the Hodge–de Rham Laplacian on the curvature 2-form $\overset{C}{\hat{R}}$ in the PGTG in spaces with torsion:

$$\Delta^{(\mathrm{HR})}\overset{C}{\hat{R}} = \left[\left(\nabla_{\mu}\delta^{\kappa}_{\nu} + 1/2T^{\kappa}_{\mu\nu}\right)\left(\nabla_{\mu}\delta^{\alpha}_{\kappa} + 1/2T^{\alpha\beta}_{\kappa} + \delta^{\alpha}_{\kappa}T^{\beta}\right)R^{\lambda}_{\sigma\alpha\beta}\right]\bar{e}_{\lambda}\otimes\bar{e}^{\sigma}\otimes\theta^{\mu}\wedge\theta^{\nu}.$$

Here, $T^{\kappa}_{\mu\nu}$ is the torsion tensor, T^{κ} is its trace and ∇_{μ} is the symbol of covariant differentiation.

If we confine ourselves to a special case of the Riemann–Cartan space, then the consequence of this expression is the statement:

The solutions of the equations of the gravitational field of the Einstein–Cartan theory in vacuum are harmonic.

The proof is based on the fact that in the Einstein–Cartan theory one of the equations of the gravitational field in vacuum is $T^{\kappa}_{[\alpha\beta]} + 2\delta^{\kappa}_{[\alpha}T_{\beta]} = 0$, and then on the use of the proved statement for general relativity.

5. Conclusion

Thus, it has been shown that both the general theory of relativity and the Poincaré-gauge theory of gravity (in the particular case of the Einstein–Cartan theory) have a property similar to the Yang–Mills and Maxwell electromagnetism theories, namely, the solutions of the field equations of these theories in vacuum are harmonic, what demonstrates the generality of the gauge theory of gravity with other gauge theories.

It is also shown that the assertion existing in the literature that the equality $\Delta^{(\mathrm{HR})}\hat{R} = 0$ holds only for solutions of the Einstein equation in vacuum of type N, and therefore can serve as a criterion for the presence of gravitational radiation, is an erroneous assertion, although it is indicated in the well-known monograph by V. D. Zakharov [6].

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Лапласиан Ходжа–де Рама и геометрический критерий для гравитационных волн

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Аннотация. Тензор кривизны $\hat{R}$ многообразия называется гармоничным, если он подчиняется условию $\Delta^{(\mathrm{HR})}\hat{R} = 0$, где $\Delta^{(\mathrm{HR})} = DD^* + D^*D$ — лапласиан Ходжа-де Рама. Доказывается, что все решения уравнений Эйнштейна в пустоте, а также все решения теории Эйнштейна–Картана в пустоте обладают гармоничной кривизной. Опровергается утверждение о том, что гармоничными являются только решения уравнений Эйнштейна типа N, описывающее гравитационное излучение.

Ключевые слова: Лапласиан Ходжа–де Рама, гармоничный тензор кривизны, гармоничные решения в пустоте уравнений Эйнштейна и уравнений теории Эйнштейна–Картана