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Convergence of the grid method for the Fredholm equation of the first kind with Tikhonov regularization

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Abstract. The paper describes a grid method for solving an ill-posed problem for the Fredholm equation of the first kind using the A. N. Tikhonov regularizer. The convergence theorem for this method was formulated and proved. A procedure for thickening grids with a simultaneous increase in digit capacity of calculations is proposed.

Key words and phrases: ill-posed problems, grid method, regularization

1. Introduction

A large number of applied tasks are ill-posed. A number of methods have been developed to solve them. Firstly, these are parametric methods in which the solution is represented as a decomposition over some basis, and the regularized equation is reduced to the problem of optimizing the coefficients of the decomposition (see, for example [1–3]). The success of this approach strongly depends on the successful choice of the basis. Such methods are difficult to study; finding estimates of accuracy and conditionality in calculations with finite digit numbers is particularly difficult. Most of the proofs are carried out for exact calculations with infinite digit capacity, i.e., without round-off errors.

Secondly, iterative methods with simple or implicit iterations [4, 5] are often used to obtain an approximate analytical solution. The number of iterations is also a regularizing parameter [6]. This looks tempting, since there is no need to introduce additional stabilizing terms and thereby increase the discrepancy. On the other hand, in the general case, iterations have to be implemented numerically. The finite-difference approximation of the corresponding quadratures introduces some systematic error in the operator and the right part. To reduce it, it is necessary to perform calculations on thickening grids.

The third approach is represented by various grid methods (finite-difference or finite-element), in which the solution is calculated in a set of discrete grid nodes, that is, essentially replaced by a piecewise constant function. In this approach, the initial problem is reduced to a system of algebraic equations that can be solved by any direct or iterative method [7, 8]. Yu. L. Gaponenko showed that finite-difference approximation makes the problem correct, i.e., self-regulation takes place [9, 10]. The study of finite element approximations (for specific applied problems) was carried out, for example, in [11, 12]. However, the proofs and convergence estimates are valid for calculations with infinite digit capacity, since they do not take into account rounding errors.

The central point of all regularizing algorithms is the justification of convergence and the evaluation of the actual accuracy, that is, the difference between the exact solution and the approximate one found. A review of the literature on this issue is given in [13]. Known a posteriori estimates are majorant and often greatly overestimate the error (up to 10 times or more). Quite often, they require specific information and solutions that are not easy to obtain in complex application tasks [14].

Another important issue is the choice of the regularization parameter. This problem is not trivial, since in most applied calculations the error level is fixed and does not tend to zero [15]. The best known solution to this question is the well-known generalized residual principle [16].

In the present paper, we describe a grid method for solving an ill-posed problem for the Fredholm equation of the first kind using the Tikhonov regularizer of the zeroth order. For this method, we formulate and prove convergence theorem which takes into account finite digit capacity of calculations. For its practical implementation, we propose procedure of simultaneous grid thickening and increase of digit capacity.

2. Method

We consider the Fredholm equation of the first kind

$$Au = f, \quad Au = \int_a^b K(y,x)u(x)dx, \quad y \in [c,d]. \tag{1}$$

A well-known technique of regularization is to add the simplest Tikhonov stabilizer to the residual [8]. This leads to the following optimization problem

$$\|Au - f\|_{L2}^2 + \alpha\|u\|_{L2}^2 \to \min. \tag{2}$$

Here, $\alpha > 0$ is a regularization parameter.

Minimizing (2) by u leads to the Euler equation. In the case of a non-self-adjoint operator A, it has the form

$$
\begin{gathered}
\int\limits_a^b Q(z,x)u(x)dx + \alpha u = F(z), \quad z \in [a,b], \\
Q(z,x) = Q(x,z) = \int\limits_c^d K(y,x)K(y,z)dy, \quad F(z) = \int\limits_c^d K(y,z)f(y)dy.
\end{gathered}
\tag{3}
$$

To solve (2), let us use convenient mesh method [17]. We introduce meshes on $x \in [a,b]$ and $y \in [c,d]$. For simplicity, they are supposed to be uniform and to have the same number of steps N. The grid steps of x and y are denoted by $h = (b-a)/N$ and $\tau = (d-c)/N$, respectively. Let us replace all integrals in (1) by quadrature rules (for definiteness, using trapezoid rule). This leads to the difference problem

$$
\begin{gathered}
\sum_{n=0}^{N} \left[(A^*A)_{k,n} + \alpha E_{k,n}\right] u_n = F_k, \quad 0 \leqslant k \leqslant N, \\
(A^*A)_{k,n} = \tau h g_n \sum_{m=0}^{M} g_m K_{m,k} K_{m,n}, \quad F_k = \tau \sum_{m=0}^{M} g_m K_{m,k} f_m.
\end{gathered}
\tag{4}
$$

Here, g are the weights of the trapezoid formula, $E_{k,n}$ is the unit matrix. The system of equations (4) is solved by some direct method.

3. Convergence

Let us formulate a few preliminary considerations.

1^o When replacing integrals with grid approximations, we introduce some error. It can be considered as systematic. This error can be estimated using the Richardson method. This method is rigorously substantiated in [18]. Recall the essence of this approach.

In sequential twofold mesh thickening, even nodes of the current mesh coincide exactly with the nodes of the previous one. In these nodes, one can directly compute the difference of solutions on the sequential grids $\delta = u_{\text{fine}} - u_{\text{coarse}}$. The error estimation takes the form

$$
r = \frac{\delta}{(2^p - 1)},
\tag{5}
$$

where p is the accuracy order of the scheme. We emphasize that this approach does not require any information on the derivatives of the exact solution and provides asymptotically precise (i.e. unimprovable) error value instead of majorant one.

The described procedure can be controlled by graphs of $\lg \|r\|_{l2}$ versus $\lg N$. If N is too small, the plot behavior is irregular. For "moderate" N, the plot is a straight line with slope $-p$. On this section of the plot, Richardson method is applicable. For excessively large N, the plot sharply passes to a horizontal line. This means that the calculation has reached round-off error background caused by finite digit capacity. Here, Richardson method is inapplicable, and one should terminate the calculations.

2º The matrix of a linear system (4) is ill-conditioned. Calculations with finite digit capacity lead to a random error associated with round-off errors. With a sufficiently small step, the calculation error becomes comparable with round-off errors and ceases to decrease with further thickening of the grids. To reduce the impact of rounding errors, one needs to increase the digit capacity of calculations. Apparently, Richtmyer was the first to point this out in the 1950s [19]. He noted that any difference scheme is incorrect in the sense that when the grid step tends to zero, it is necessary to increase the digit capacity of calculations.

At the same time, theorems on regularizing properties are usually proved for exact calculations (i.e., with infinite digit capacity). However, real calculations are carried out on finite round-off errors. It often turns out that in ill-conditioned problems, computer round-off errors can become predominant.

3º The use of a regularizer improves the conditionality of the linear system matrix. Therefore, increasing α reduces the random error (for calculations with fixed bit depth). However, the regularizer itself introduces a systematic error in the problem, which increases with increasing α.

Based on these suggestive considerations, we formulate the convergence theorem of the grid method (4). As far as we know, it is new.

Theorem 1. *For any precision $\varepsilon > 0$, there exist $\alpha_0 > 0$, step h_0 and digit capacity K_0 such that for $h < h_0$, $K > K_0$ and $\alpha = \alpha_0$ the error is less than ε.*

Proof. The proof consists of 3 stages. We write down the regularized Fredholm equation. For it, according to Tikhonov's fundamental theorem, there is a required value of α_0.

By virtue of the Ryabenky–Fillipov theorems, there is such a h_0 that provides a systematic approximation error that does not exceed the required one for calculations with infinite digit capacity.

Since, for the selected grid step h, the conditionality of the linear system is known, then there is such a digit capacity that provides the required smallness of the random error. The theorem is proved. □

4. Calculation procedure

For the practical implementation of this theorem, the following algorithm is proposed. Let us set some K and α and perform the calculation with grid thickening. On each grid, we calculate the error estimate using the Richardson method. We thicken the grids until this estimate stops decreasing. Denote the last solution obtained as the ***limiting*** one.

Let us perform such calculations for a wide range of α values. The dependence of the true error of the limiting solution (i.e., the difference between numerical and exact solutions) on α has the following qualitative form. For $\alpha = 0$, the error is very large due to poor conditionality of the matrix A^*A. For small α, the random error is predominant, and the systematic error is negligible. As α increases, the random error decreases, and the systematic error, on the contrary, increases due to the term $\sim \alpha\|u\|^2$ in the regularizer. With some α, the random and systematic errors become equal. This α corresponds to the best achievable accuracy at the selected bit depth.

The value of the random error is estimated as the product of the unit rounding error δ_0 by the condition number κ of a linear system (4). For calculations with 64-bit numbers, we have $\delta_0 = 10^{-16.2}$. To estimate κ, it is advisable to use the angular conditionality number [20]. As noted above, the systematic error consists of the grid approximation error (which is calculated using the Richardson method) and the regularizer contribution. In the zeroth approximation, these contributions can be considered independent. Therefore, according to the rules of statistics, the total value of the systematic error can be estimated as $\sqrt{\|r\|^2 + \alpha^2\|u\|^2}$.

As the final one, we choose such a α, in which the estimates of random and systematic error are equal. If the obtained accuracy is unsatisfactory, one should increase the digit capacity and repeat the described calculations.

As far as we know, such calculation procedures with simultaneous thickening of grids and increasing digit capacity have not been proposed before.

5. Conclusion

Let us discuss possible generalizations of the proposed approaches. Firstly, the convergence theorem admits generalization to the case when the difference scheme is compiled not for a regularized problem, but for an initial ill-posed one. The absence of a regularizer reduces the systematic error. However, obviously, a significantly larger number of digits is required, which increases the complexity of the calculation.

Secondly, it is also advisable to use the procedure of thickening grids with a simultaneous increase in digit capacity for the numerical solution of formally correct, but ill-conditioned problems. Examples are stiff Cauchy problems with contrast structures. It is easy to construct a problem in which, when calculating 64-bit numbers, there is not a single correct sign in the answer [17]. Note that ill-conditionality and round-off errors are one of the important factors limiting the applicability of grid methods. Therefore, the relaxation of this restriction is of great practical interest.

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Сходимость сеточного метода для уравнения Фредгольма первого рода с регуляризацией по Тихонову

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Аннотация. В статье описан сеточный метод решения некорректной задачи для уравнения Фредгольма первого рода с использованием регуляризатора А. Н. Тихонова. Сформулирована и доказана теорема о сходимости этого метода. Для её практической реализации предложена процедура сгущения сеток с одновременным увеличением разрядности вычислений.

Ключевые слова: некорректные задачи, сеточный метод, регуляризация