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Construction, stochastization and computer study of dynamic population models “two competitors – two migration areas”

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Abstract. When studying deterministic and stochastic population models, the actual problems are the formalization of processes, taking into account new effects caused by the interaction of species, and the development of computer research methods. Computer research methods make it possible to analyze the trajectories of multidimensional population systems. We consider the “two competitors – two migration areas” model, which takes into account intraspecific and interspecific competition in two populations, as well as bidirectional migration of both populations. For this model, we take into account the variability of the reproduction rates of species. A formalized description of the four-dimensional model “two competitors – two migration areas” and its modifications is proposed. Using the implementation of the evolutionary algorithm, a set of parameters is obtained that ensure the coexistence of populations under conditions of competition between two species in the main area, taking into account the migration of these species. Taking into account the obtained set of parameters, a positive stationary state is found. Two-dimensional and three-dimensional projections of phase portraits are constructed. Stochastization of the model “two competitors – two migration areas” is carried out based on the method of self-consistent one-step models constructing. The Fokker–Planck equations are used to describe the structure of the model. A transition to a four-dimensional stochastic differential equation in the Langevin form is performed. To carry out numerical experiments, a specialized software package is used to construct and study stochastic models, and a computer program based on differential evolution is developed. Algorithms for generating trajectories of the Wiener process and multipoint distributions and modifications of the Runge–Kutta method are used. In the deterministic and stochastic cases, the dynamics of the trajectories of population-migration systems is studied. A comparative analysis of deterministic and stochastic models is carried out. The results can be used in modeling of different classes of dynamic systems.

Key words and phrases: population dynamics models, stochastic differential equations, one-step processes, stochastization, competition, migration, trajectory dynamics, projections of phase portraits, computer modeling, software package

1. Introduction

The classical Lotka–Volterra models [1, 2] are further developed in numerous papers by researchers [3–8]. Significant progress is associated with the analysis of dynamic models of ecological systems using the methods of the theory of stability of solutions of differential equations and optimization theory [3–6, 9–11]. It should be noted that when studying population models, the transition from the deterministic to the stochastic case is of great theoretical and applied interest [12–15].

Population dynamic models are characterized by the fact that when describing them, it is necessary to take into account various types of interaction in the population community, for example, intraspecific competition, interspecific competition, trophic interactions, migration, mutualism [16–18]. For example, research is being conducted related to the study of the properties of multidimensional ecological and demographic systems, taking into account competition and migration flows [19–22]. As the results show, the impact of migration can be significant, and the presence of migration flows leads to the emergence of new qualitative effects. The presence of migration flows in a population system is associated with an adaptive change in the behavior of an organism under changing environmental conditions, in particular, with a deterioration in the epidemiological situation or with an increase in population densities [23]. When constructing models, migration mechanisms are described using linear and nonlinear functions [15, 24–26]. The stability and qualitative behavior of population-migration models are considered in [18, 24, 26–29] and other papers. Despite a number of interesting results in the direction of studying systems with migration flows, there is a need to construct and research new models with migration.

As is known, one-dimensional Fokker–Planck equations are used in the construction of Gaussian stochastic models of small dimension. For multidimensional models, the simplest linear models with additive noise are most often used, however, this approach does not fully take into account stochastic processes in the system. A promising direction is stochastic modeling of dynamic systems based on the method of constructing self-consistent one-step models [30–32]. Using this method, we can perform an algorithmic transition to a stochastic model and evaluate the influence of stochastics on the qualitative properties of the model. This assessment is performed through a comparative analysis of deterministic and stochastic models with selected sets of parameters. When studying high-dimensional models, the choice of parameters can be carried out by applying evolutionary algorithms [12, 33–35]. Various systems of population dynamics (with competition, mutualism, migration) based on self-consistent models are considered in [12, 14, 28] and other papers.

Researchers consider various generalizations and modifications of the classical Lotka–Volterra models in the direction of increasing the dimension and constructing non-deterministic models. When considering such models, there is a need for computer research, taking into account the capabilities of high-level languages and applied mathematical packages [36–39]. Numerical analysis of behavior and computer studies of the dynamics of trajectories are associated, among other things, with new problems in the study of nonlinear processes, taking into account the processing of large data arrays under uncertainty. A software package is developed for stochastic modeling of various dynamic systems based on the method of constructing self-consistent one-step models [30, 31]. For the controlled case, a set of programs is proposed that combines randomization, optimization and machine learning [12].

Modeling of population-migration systems is carried out using various software that have a fairly effective set of tools for constructing computer models and conducting computational experiments [38, 40]. The use of applied mathematical packages and high-level programming languages makes it possible to study multidimensional population systems taking into account different types of intraspecific and interspecific interactions, as well as taking into account the variation of parameters and variables.

The three-dimensional model “predator-prey-one migration area” is considered in [29]. Four-dimensional population models with competition and one area of migration are studied in [20, 41]. This article is devoted to the study of such a four-dimensional population model of the type “two competitors – two migration areas”, which takes into account changes in the reproduction rates of populations.

Section 2 of the paper considers the construction of the “two competitors – two migration areas” model with bidirectional migration (to two refuges) and its modifications. In particular, we offer a description of the model, in which the reproduction rate of population growth are different without varying the parameters of competition and migration. In Section 3, search for model parameters using an evolutionary algorithm is carried out. A study of a deterministic four-dimensional model is carried out, two-dimensional and three-dimensional projections of phase portraits are constructed. In Section 4, stochastic models “two competitors – two migration areas” are constructed using the method of constructing self-consistent stochastic models. In Section 5 the dynamics of trajectories for deterministic and stochastic models are studied. The results of computer experiments are presented and the interpretation of these results is given taking into account the comparison of stochastic and deterministic models. A software package developed in Python using the NumPy, SymPy, SciPy libraries is used as a tool for studying models.

2. Description of the deterministic model “two competitors – two migration areas” and its modifications

One of the basic population-migration models, taking into account competition and migration flows, is a three-dimensional model that describes the dynamics of two interrelated species. According to this model, the first species competes with the second species in the first area, taking into account the migration of the first species to the second area [19]. Four-dimensional generalizations of this population-migration model are studied in [19, 21, 22, 42] and in other papers.

Next, we describe a four-dimensional model that takes into account the influence of interspecies competition in two populations with bidirectional migration of both populations. This model is given by a system of nonlinear differential equations of the form

$$\begin{aligned}
\dot{x}_1 &= a_1x_1 - p_{11}x_1^2 - p_{13}x_1x_3 + \beta x_2 - \gamma x_1,\\
\dot{x}_2 &= a_2x_2 - p_{22}x_2^2 + \gamma x_1 - \beta x_2,\\
\dot{x}_3 &= a_3x_3 - p_{33}x_3^2 - p_{31}x_1x_3 + \varepsilon x_4 - \delta x_3,\\
\dot{x}_4 &= a_4x_4 - p_{44}x_4^2 + \delta x_3 - \varepsilon x_4.
\end{aligned} \tag{1}$$

where x_1 is density of a competing population of the first species in the first area; x_2 is a population density of the first species in the second area (in the first refuge); x_3 is density of the competing population of the second species in the first area; x_4 is a population density of the second species in the third area (in the second refuge); a_i, $i = 1, 2, 3, 4$, are natural growth coefficients; p_{13}, p_{31} are coefficients of interspecific competition; p_{11}, p_{22}, p_{33}, p_{44} are coefficients of intraspecific competition; β, γ are coefficients of migration of a species between the first and second areas, with the second area is a refuge; δ, ε are coefficients of species migration between the first and third areas, with the third area is a refuge.

We consider a particular case of the model (1), when $p_{13} = p_{31} = r$, $p_{11} = p_{22} = p_{33} = p_{44} = p$, $\beta = \gamma$, $\varepsilon = \delta$, model (1) takes the form:

$$\begin{aligned}
\dot{x}_1 &= a_1x_1 - px_1^2 - rx_1x_3 + \beta x_2 - \beta x_1,\\
\dot{x}_2 &= a_2x_2 - px_2^2 + \beta x_1 - \beta x_2,\\
\dot{x}_3 &= a_3x_3 - px_3^2 - rx_1x_3 + \delta x_4 - \delta x_3,\\
\dot{x}_4 &= a_4x_4 - px_4^2 + \delta x_3 - \delta x_4.
\end{aligned} \tag{2}$$

Analysis of models (1), (2) involves finding of stationary states corresponding to stationary population densities. The search for most stationary states for models (1),(2) in an analytical form is difficult due to the dimension of the models and a large number of parameters. However, in this situation, it is possible to search for particular sets of model parameters using evolutionary numerical optimization algorithms. Note that for a particular case of model (1), when $p_{13} = p_{31} = r$, $p_{11} = p_{22} = p_{33} = p_{44} = p$, $a_1 = a_2 = a_3 = a_4 = a$, $\beta = \gamma$, $\varepsilon = \delta$, the problem of finding parameters that ensure the mode of population coexistence is studied in [21].

3. Analysis of stationary states and construction of projections of phase portraits for a four-dimensional population-migration model (2)

In this section of the paper, we develop the approach proposed in [35] to the optimization search for the parameters of population models. The approach considered in this paper makes it possible to obtain such sets of model parameters that ensure the achievement of the extremum of the quality criterion. We propose to use a quality criterion by analogy with [35] to study model (2).

We consider the optimization problem of species coevolution in system (2). Within the frames of this problem, we will track the change in the coefficients a_1, a_2, a_3, a_4, p, r, β, δ. These coefficients are limited by some positive values contained in the parametric set A of system (2). Namely, one can write

$$(a_1, a_2, a_3, a_4, p, r, \beta, \delta) \in A. \tag{3}$$

One of the variants for the optimality conditions has the form

$$\int_0^{t_1} x_i(t)dt \to \max, \quad i = 1, 2, 3, 4, \tag{4}$$

where t_1 is the boundary of the studied time interval. However, in model (2), to take into account the requirement for the coexistence of two populations in three areas (in an area with interspecific competition and in two refuges) we will use instead of condition (4) an optimality condition of the form

$$\int_{t_0}^{t_1} x_1(t)x_2(t)x_3(t)x_4(t)dt \to \max, \tag{5}$$

where t_0 is the expected time for system (2) to reach the stationary mode. In the case of extinction of one of the two species (both in an area with interspecific competition and in two refuges), the integrand vanishes.

To solve the optimization problem (2),(5), we use the differential evolution algorithm. For the convenience of computational procedures, we write condition (5) in the form

$$\int_{t_0}^{t_1} (x_1(t)x_2(t)x_3(t)x_4(t)dt)^{-1} \to \min, \tag{6}$$

Estimation of fulfillment of condition (6) is the objective function (loss function) for the optimization algorithm. If this condition is met, then the algorithm terminates. Algorithm has the form:

```
def fit_evo(w):
        plane,time=eval_eco_4d(w)
        p = np.array(plane)
        p1 = p[:,0]; p2 = p[:,1];
        p3 = p[:,2]; p4 = p[:,3]
        px = p1*p2*p3*p4
        error = np.trapz(px[-30:])
        if error <= 0: return 1000
        else: return error**(-1)
```

For stable operation of the algorithm, negative and zero results are eliminated by assigning a fixed value of 1000. Optimization is carried out using the differential evolution method implemented in the scientific computing library SciPy of Python language. The connection of differential evolution is carried out as follows:

```
scipy.optimize.differential_evolution(parametres).
```

With the help of a program created in the Python language, a computational experiment is carried out to select the parameters of the model (2). The parameter values are limited by the interval [0.3, 10], the maximum number of iterations of the differential evolution method is 100. The code snippet has the form:

```
bounds = list([(0.3,10) for i in range(8)])
x0 = list([0 for i in range(8)])
res = differential_evolution(fit_evo, bounds,
maxiter = 100, workers = 1, disp = True)
print(res)
```

The RK45 module (Runge–Kutta method) from SciPy is connected as a solver. Model (2) is defined by the following class:

```
class model_eco_4d(ode_model):
        def eq(self, t, y):
                [a1,a2,a3,a4], p, r, beta, delta =
                map(self.args.get, "a p r beta delta".split())
                x0,x1,x2,x3 = y
                dx1=a1*x0-p*x0**2-r*x0*x2+beta*x1-beta*x0
                dx2=a2*x1-p*x1**2+beta*x0-beta*x1
                dx3=a3*x2-p*x2**2-r*x0*x2+delta*x3-delta*x2
                dx4=a4*x3-p*x3**2+delta*x2-delta*x3
                return [dx1,dx2,dx3,dx4]
        def regulate(self, P):
                pass
        @property
        def current_time(self): return self.states.t
```

The solving of the optimization problem of searching for parameters of model (2), taking into account condition (6), made it possible to conduct a computational experiment and find the corresponding set of parameters under the following initial conditions $(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.5, 0.5, 1, 7)$. The specified set of parameters has the form

$$\begin{aligned}(a_1, a_2, a_3, a_4, p, r, \beta, \delta) = (&5.674e+00, 8.693e+00, 9.045e+00,\\ &9.637e+00, 3.595e-01, 7.522e-01, 3.252e+00, 7.500e-01). \quad (7)\end{aligned}$$

Taking into account rounding to two decimal places, the set of parameters will take the form: 5.67, 8.70, 9.05, 9.64, 0.36, 0.75, 3.25, 0.75. Using the found set of parameters, a positive stationary state is obtained: $x_1 = 5.09$, $x_2 = 17.73$, $x_3 = 15.84$, $x_4 = 25.99$. The considered approach to the search for stationary states is of interest due to the fact that for many high-dimensional population systems (including four-dimensional systems (1)–(2)) finding stationary states in a general form causes difficulties even when using computing packages.

Next, we consider examples of the construction of phase portraits projections for model (2). The projection of the phase portrait on the plane (x_1, x_2), taking into account $x_3 = 15.84$, $x_4 = 25.99$, is shown in the figure 1. The projection of the phase portrait in space (x_1, x_2, x_4) with regard to $x_3 = 15.84$ for model (2) is shown in the figure 2.

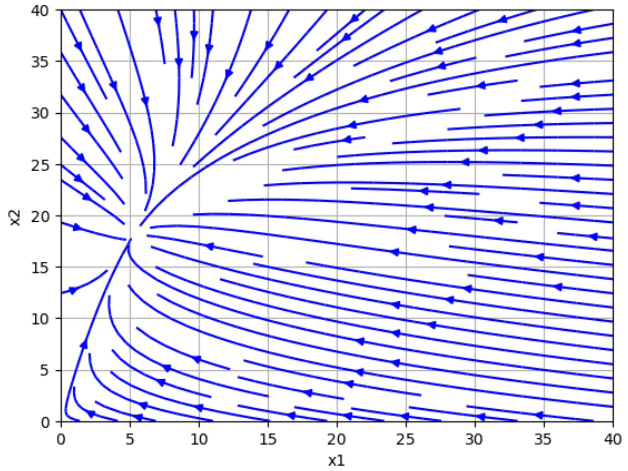


Figure 1. Projection of the phase portrait on the plane (x_1, x_2) for system (2) at $(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.5, 0.5, 1, 7)$, $a_1 = 5.67$, $a_2 = 8.70$, $a_3 = 9.05$, $a_4 = 9.64$, $p = 0.36$, $r = 0.75$, $\beta = 3.25$, $\delta = 0.75$

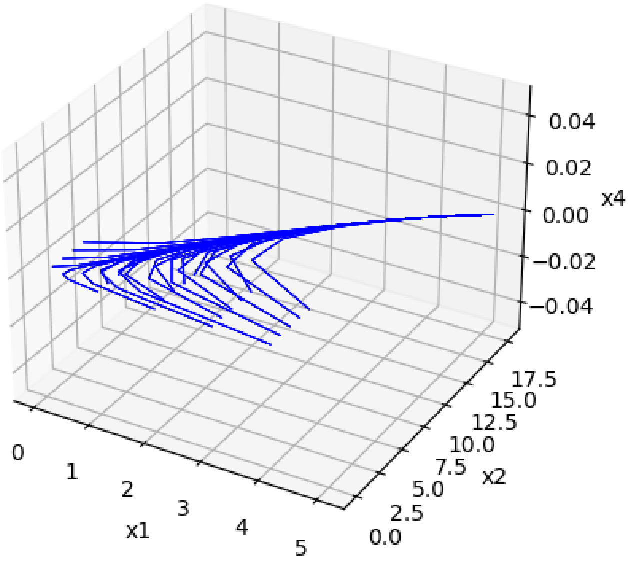


Figure 2. Projection of the phase portrait in space (x_1, x_2, x_4) for system (2) at $(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.5, 0.5, 1, 7)$, $a_1 = 5.67$, $a_2 = 8.70$, $a_3 = 9.05$, $a_4 = 9.64$, $p = 0.36$, $r = 0.75$, $\beta = 3.25$, $\delta = 0.75$

The projection of the phase portrait on the plane (x_1, x_3), taking into account $x_2 = 17.73$, $x_4 = 25.99$, is shown in figure 3. The projection of the phase portrait in space (x_1, x_3, x_4) with regard to $x_2 = 17.73$ for model (2) is shown in the figure 4.

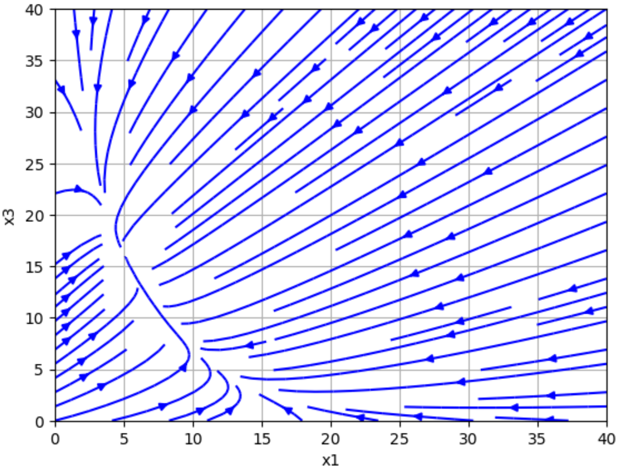


Figure 3. Projection of the phase portrait on the plane (x_1, x_3) for system (2) at $(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.5, 0.5, 1, 7)$, $a_1 = 5.67$, $a_2 = 8.70$, $a_3 = 9.05$, $a_4 = 9.64$, $p = 0.36$, $r = 0.75$, $\beta = 3.25$, $\delta = 0.75$

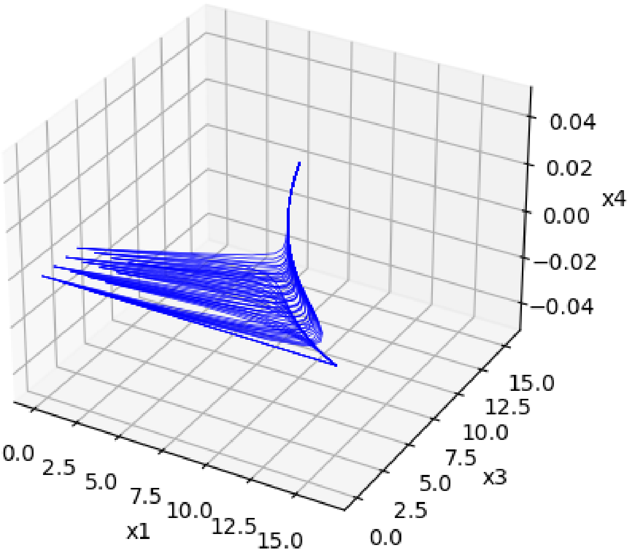


Figure 4. Projection of the phase portrait in space (x_1, x_3, x_4) for system (2) at $(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.5, 0.5, 1, 7)$, $a_1 = 5.67$, $a_2 = 8.70$, $a_3 = 9.05$, $a_4 = 9.64$, $p = 0.36$, $r = 0.75$, $\beta = 3.25$, $\delta = 0.75$

The projection of the phase portrait on the plane (x_2, x_3), taking into account $x_1 = 17.73$, $x_4 = 25.99$, is shown in figure 5. The projection of the phase portrait in space (x_2, x_3, x_4), taking into account $x_2 = 17.73$ for model (2), is shown in the figure 6.

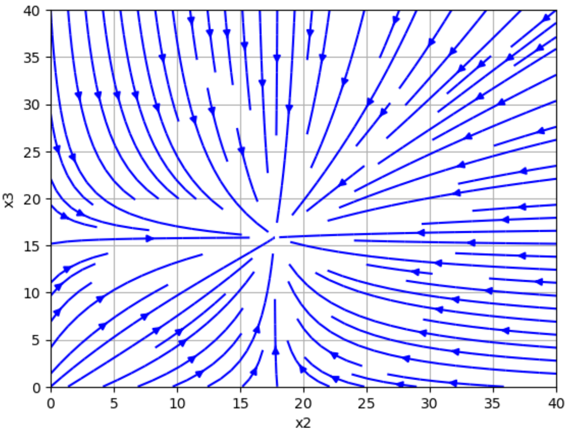


Figure 5. Projection of the phase portrait on the plane (x_2, x_3) for system (2) at $(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.5, 0.5, 1, 7)$, $a_1 = 5.67$, $a_2 = 8.70$, $a_3 = 9.05$, $a_4 = 9.64$, $p = 0.36$, $r = 0.75$, $\beta = 3.25$, $\delta = 0.75$

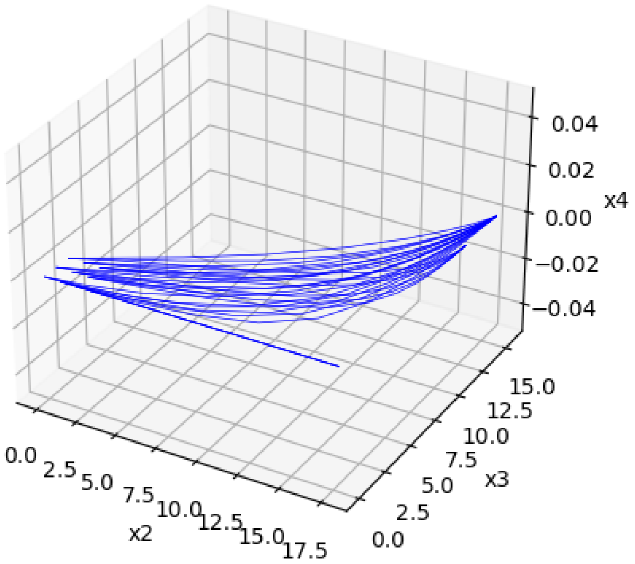


Figure 6. Projection of the phase portrait in space (x_2, x_3, x_4) for system (2) at $(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.5, 0.5, 1, 7)$, $a_1 = 5.67$, $a_2 = 8.70$, $a_3 = 9.05$, $a_4 = 9.64$, $p = 0.36$, $r = 0.75$, $\beta = 3.25$, $\delta = 0.75$

According to figures 1–6 the stationary state $x_1 = 5.09$, $x_2 = 17.73$, $x_3 = 15.84$, $x_4 = 25.99$ has the character of a stable node. In this section, for clarity, separate variants for two-dimensional and three-dimensional projections are given taking into account the fact that a complete set of projections can be built in a similar way. The construction of phase portraits projections made it possible to analyze the nature of the dependence of phase variables on each other.

4. Construction of stochastic models "two competitors – two migration areas"

For further analysis of the qualitative properties of the models, we study the influence of stochastics introduction. For this purpose, it is proposed to develop a stochastic model by constructing self-consistent stochastic models [13, 30, 32, 43, 44]. This method involves, in the course of mathematical transformations, obtaining the coefficients of the Fokker–Planck equation from the interaction scheme, which allows implementing a symbolic record of all possible interactions in the system. The basic steps of this method are presented in the description of algorithm 1.

Algorithm 1: Obtaining a model from a system description

```
Initial parameters: description of the system.
Result: Deterministic and stochastic models (ODE and SDE systems).
begin
    1. Adding interactions to the model.
    2. Construction of the interaction scheme of the model.
    3. Construction of system state operators from the interaction
       scheme ($M$ and $N$).
    4. Construction of an operator for changing the state of the system
       ($R$).
    5. Construction of transition intensities ($s$).
    6. Construction of the coefficients for the Fokker–Planck equation
       ($A$ and $B$).
    7. Construction of the SDE.
```

This algorithm is implemented in the Python programming language using the NumPy and SciPy libraries and it is described in [31]. The software package used in this paper makes it possible to construct a stochastic model of a dynamic system from its description, to construct the corresponding deterministic model, to obtain numerical solutions of ODEs and SDEs, as well as a graphical representation of the solutions.

When implementing algorithm 1, the input data is not an interaction scheme, but a description of the interactions occurring in the system. For this, the `PopModel` class is used. Calling the `PopModel(n)` constructor, where `n` is the dimension of the system, creates a class object that is a formal model of the system. The resulting object contains a description of all considered system interactions. The `adder()` method of the `PopModel` class allows to add the main types of population interactions to the model, such as natural reproduction, natural death, competition, symbiosis, predator-prey relationships.

For the generalized model (1), we define a class object:

```
model_1 = pm.PopModel(4)
```

and add a description of interactions

```
# natural reproduction
model_1.adder(1,1,0,"a_1")
model_1.adder(1,2,0,"a_2")
model_1.adder(1,3,0,"a_3")
model_1.adder(1,4,0,"a_4")

# Intraspecific competition
model_1.adder(5,1,0,"p_11")
model_1.adder(5,2,0,"p_22")
model_1.adder(5,3,0,"p_33")
model_1.adder(5,4,0,"p_44")

# Interspecies competition
model_1.adder(6,3,1,"p_13")
model_1.adder(6,1,3,"p_31")

# Migration
model_1.adder(7,1,2,"γ")
model_1.adder(7,2,1,"β")
model_1.adder(7,3,4,"ϵ")
model_1.adder(7,4,3,"δ")
```

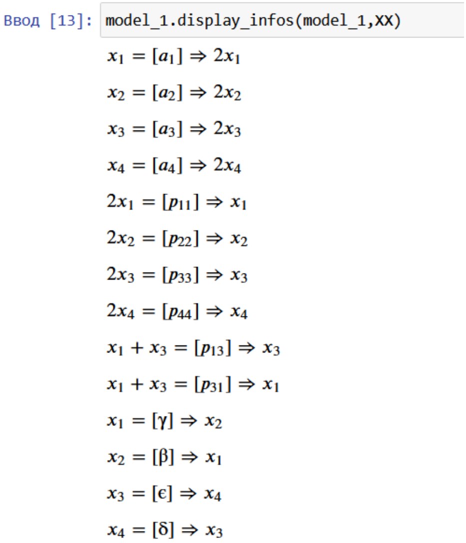
The first four lines of interactions description correspond to natural reproduction, lines 5–8 describe intraspecific competition, lines 9 and 10 are responsible for interspecific competition, and lines 11–14 describe population migration.

The `display_infos` method allows to display the interaction diagram. When using the Jupiter interactive shell, the output will have the form shown in the figure 7.

The IstoDE.py module allows to get the coefficients of the Fokker–Planck equation from the interaction diagram. The basic functions of this module are the `drift_vector` and `diffusion_matrix` functions. The first function is designed to obtain the drift vector A in the Fokker–Planck equation. With the help of the second function, we can get the diffusion matrix in the Fokker–Planck equation. As input parameters, these two functions take the state vector of the system, coefficients, matrices N and M.

In the figure 8 shows the result of the derivation of functions for obtaining the coefficients of the Fokker–Planck equation in relation to the interaction scheme in the figure 7.

Model (2) can be obtained from the constructed model by redefining the interaction coefficients as follows: $p_{13} = p_{31} = r$, $p_{11} = p_{22} = p_{33} = p_{44} = p$, $\beta = \gamma$, $\varepsilon = \delta$. In figure 9 shows the output of the `drift_vector` and `diffusion_matrix` functions for model (2).



```
Ввод [13]: model_1.display_infos(model_1,XX)
```

$x_1 = [a_1] \Rightarrow 2x_1$

$x_2 = [a_2] \Rightarrow 2x_2$

$x_3 = [a_3] \Rightarrow 2x_3$

$x_4 = [a_4] \Rightarrow 2x_4$

$2x_1 = [p_{11}] \Rightarrow x_1$

$2x_2 = [p_{22}] \Rightarrow x_2$

$2x_3 = [p_{33}] \Rightarrow x_3$

$2x_4 = [p_{44}] \Rightarrow x_4$

$x_1 + x_3 = [p_{13}] \Rightarrow x_3$

$x_1 + x_3 = [p_{31}] \Rightarrow x_1$

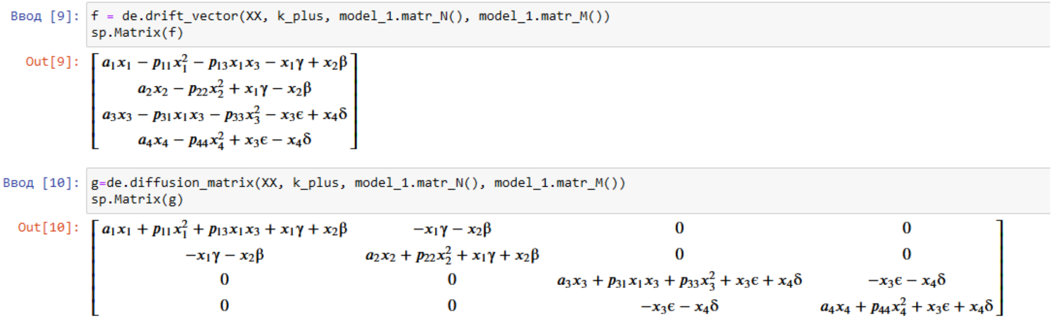
$x_1 = [\gamma] \Rightarrow x_2$

$x_2 = [\beta] \Rightarrow x_1$

$x_3 = [\epsilon] \Rightarrow x_4$

$x_4 = [\delta] \Rightarrow x_3$

Figure 7. The output of the `display_infos` method



```
Ввод [9]: f = de.drift_vector(XX, k_plus, model_1.matr_N(), model_1.matr_M())
          sp.Matrix(f)
```

Out[9]:

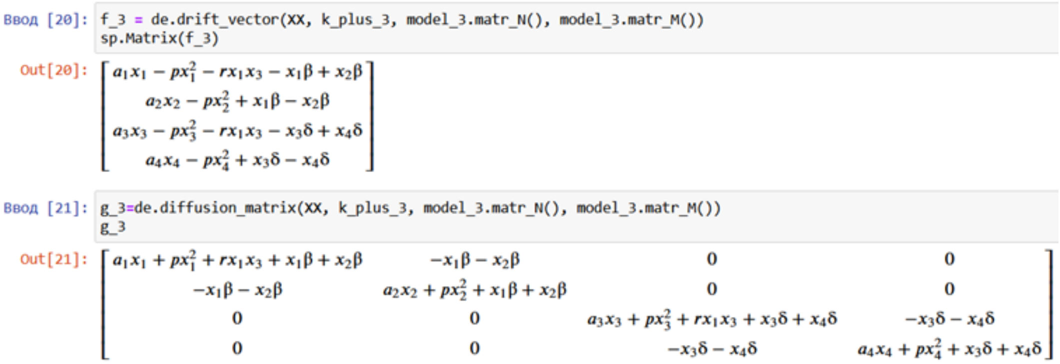
$$\begin{bmatrix} a_1x_1 - p_{11}x_1^2 - p_{13}x_1x_3 - x_1\gamma + x_2\beta \\ a_2x_2 - p_{22}x_2^2 + x_1\gamma - x_2\beta \\ a_3x_3 - p_{31}x_1x_3 - p_{33}x_3^2 - x_3\epsilon + x_4\delta \\ a_4x_4 - p_{44}x_4^2 + x_3\epsilon - x_4\delta \end{bmatrix}$$

```
Ввод [10]: g=de.diffusion_matrix(XX, k_plus, model_1.matr_N(), model_1.matr_M())
           sp.Matrix(g)
```

Out[10]:

$$\begin{bmatrix} a_1x_1 + p_{11}x_1^2 + p_{13}x_1x_3 + x_1\gamma + x_2\beta & -x_1\gamma - x_2\beta & 0 & 0 \\ -x_1\gamma - x_2\beta & a_2x_2 + p_{22}x_2^2 + x_1\gamma + x_2\beta & 0 & 0 \\ 0 & 0 & a_3x_3 + p_{31}x_1x_3 + p_{33}x_3^2 + x_3\epsilon + x_4\delta & -x_3\epsilon - x_4\delta \\ 0 & 0 & -x_3\epsilon - x_4\delta & a_4x_4 + p_{44}x_4^2 + x_3\epsilon + x_4\delta \end{bmatrix}$$

Figure 8. The output of `drift_vector` and `diffusion_matrix` functions for model (1)



```
Ввод [20]: f_3 = de.drift_vector(XX, k_plus_3, model_3.matr_N(), model_3.matr_M())
           sp.Matrix(f_3)
```

Out[20]:

$$\begin{bmatrix} a_1x_1 - px_1^2 - rx_1x_3 - x_1\beta + x_2\beta \\ a_2x_2 - px_2^2 + x_1\beta - x_2\beta \\ a_3x_3 - px_3^2 - rx_1x_3 - x_3\delta + x_4\delta \\ a_4x_4 - px_4^2 + x_3\delta - x_4\delta \end{bmatrix}$$

```
Ввод [21]: g_3=de.diffusion_matrix(XX, k_plus_3, model_3.matr_N(), model_3.matr_M())
           g_3
```

Out[21]:

$$\begin{bmatrix} a_1x_1 + px_1^2 + rx_1x_3 + x_1\beta + x_2\beta & -x_1\beta - x_2\beta & 0 & 0 \\ -x_1\beta - x_2\beta & a_2x_2 + px_2^2 + x_1\beta + x_2\beta & 0 & 0 \\ 0 & 0 & a_3x_3 + px_3^2 + rx_1x_3 + x_3\delta + x_4\delta & -x_3\delta - x_4\delta \\ 0 & 0 & -x_3\delta - x_4\delta & a_4x_4 + px_4^2 + x_3\delta + x_4\delta \end{bmatrix}$$

Figure 9. The output of `drift_vector` and `diffusion_matrix` functions for model (2)

According to figures 8, 9, the drift vectors are fully consistent with the right parts of the systems of equations for models (1) and (2), respectively. This circumstance indicates that the drift vector can be used to study the deterministic behavior of the system. The result of the work of the `IstoDE.py` module of the software package is the construction of a stochastic and deterministic model of population dynamics for further study of trajectories.

5. Results of computer experiments

In this paper, for the numerical solution of systems of ordinary differential equations, a software implementation of the standard Runge–Kutta methods of fourth order is used. To solve the corresponding stochastic differential equations, we use a specially developed library, a detailed description of which is contained in [30, 31].

The results of numerical experiments, taking into account the parameters obtained in section 3, for the constructed deterministic model (2) and the corresponding stochastic model are shown in the figure 10. We consider the following initial conditions: $x_1(0) = 0.5$, $x_2(0) = 0.5$, $x_3(0) = 1$, $x_4(0) = 7$.

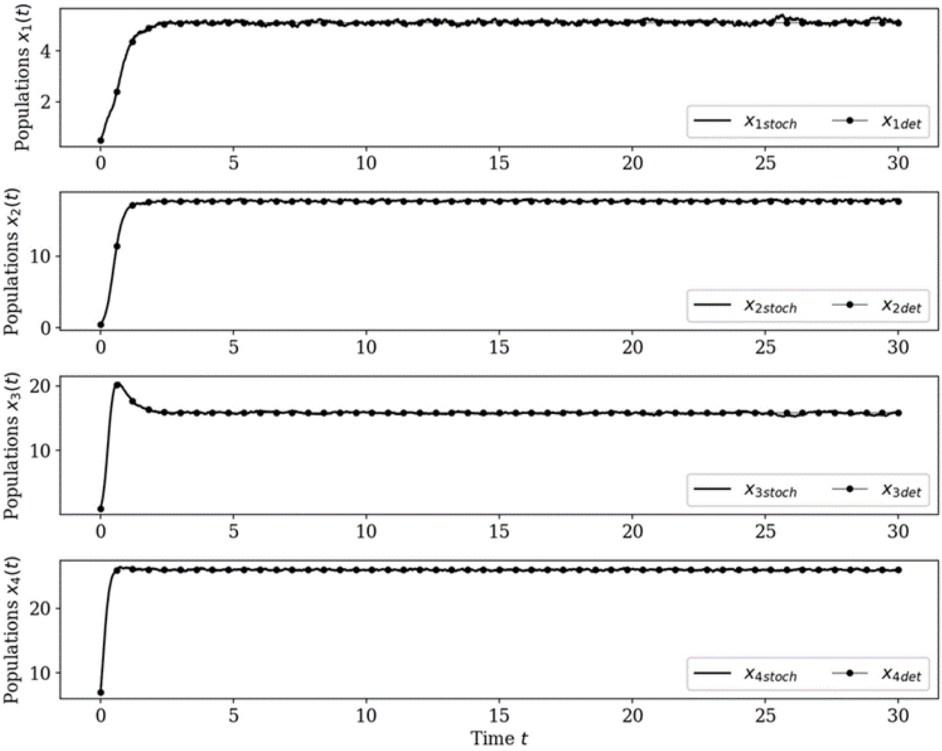


Figure 10. Visualization of the numerical solutions for model (2) and its corresponding stochastic model with parameters $a_1 = 5.67$, $a_2 = 8.70$, $a_3 = 9.05$, $a_4 = 9.64$, $p = 0.36$, $r = 0.75$, $\beta = 3.25$, $\delta = 0.75$ and with initial conditions $(x_1(0), x_2(0), x_3(0), x_4(0)) = (0.5, 0.5, 1, 7)$

According to figure 10, the introduction of stochastics has insignificant influence on the behavior of the model. As in the deterministic case, the trajectories of solutions of stochastic differential equations corresponding to model (2) are characterized by reaching the stationary mode.

6. Conclusion

The paper uses such an approach to the study of four-dimensional population-migration models, which is based on the implementation of evolutionary algorithms for searching for parameters, of the method of constructing self-consistent stochastic models and of modified methods for the numerical solution of finite-dimensional differential systems. The solution of the optimization problem by the method of differential evolution made it possible to find the optimal parameters of the «two competitors – two migration areas» model with competition of two species in the basic area and with migration to two refuges. For this model, an approximate positive stationary state is found that corresponds to the obtained set of parameters.

In this paper, based on the methods used, it is possible to construct new stochastic models of population dynamics, taking into account competition and bidirectional migration. The implementation of the stochastization algorithm for model (2) according to the given interaction schemes made it possible to analyze the trajectory dynamics of the stochastic model in comparison with the deterministic model.

As directions for further research, one can indicate the construction of new modifications of population-migration models based on model (1), the transition to the description and study of controlled ecological systems, as well as the identification of such sets of parameters that lead to a significant difference in the dynamics deterministic and stochastic models.

References

[1] V. Volterra, "Fluctuations in the abundance of a species considered mathematically," *Nature*, no. 118, pp. 558–560, 1926. DOI: `10.1038/118558a0`.

[2] A. J. Lotka, *Elements of physical biology.* Baltimore, MD, USA: Williams and Wilkins Company, 1925.

[3] A. D. Bazykin, *Nonlinear dynamics of interacting populations [Nelineynaya dinamika vzaimodeystvuyushchikh populyatsiy].* Moscow–Izhevsk: Institute of Computer Research, 2003, in Russian.

[4] A. Y. Aleksandrov, A. V. Platonov, V. N. Starkov, and N. A. Stepenko, *Study of mathematical modeling and sustainability of biological societies [Matematicheskoye modelirovaniye i issledovaniye ustoychivykh biologicheskikh soobshchestv].* St. Petersburg: Lan, 2017, in Russian.

[5] P. Turchin, *Complex population dynamics.* Princeton: Princeton University Press, 2013.

[6] Y. A. Pykh, *Generalized Lotka–Volterra systems: theory and applications [Obobshchennyye sistemy Lotki–Vol'terra: teoriya i prilozheniya].* St. Petersburg: SPbGIPSR, 2017, in Russian.

[7] L. Stucchi, J. Pastor, J. Garcia-Algarra, and J. Galeano, "A general model of population dynamics accounting for multiple kinds of interaction," *Complexity*, vol. 2020, p. 7 961 327, 2020. DOI: `10.1155/2020/7961327`.

[8] J. S. Link, F. Pranovi, and S. Libralato, "Simulations and interpretations of cumulative trophic theory," *Ecological Modelling*, vol. 463, p. 109 800, 2022. DOI: `10.1016/j.ecolmodel.2021.109800`.

[9] A. A. Shestakov, *Generalized direct method of Lyapunova for systems with distributed parameters [Obobshchennyy pryamoy metod Lyapunova dlya sistem s raspredelennymi parametrami].* Moscow: URSS, 2007, in Russian.

[10] A. I. Moskalenko, *Methods of nonlinear mappings in optimal control. Theory and applications to models of natural systems [Metody nelineynykh otobrazheniy v optimal'nom upravlenii (teoriya i prilozheniya k modelyam prirodnykh sistem)].* Novosibirsk: Nauka, 1983, in Russian.

[11] O. V. Druzhinina and O. N. Masina, *Methods for analyzing the stability of dynamic intelligent control systems [Metody analiza ustoychivosti dinamicheskikh sistem intellektnogo upravleniya].* Moscow: URSS, 2016, in Russian.

[12] A. V. Demidova, O. V. Druzhinina, O. N. Masina, and A. A. Petrov, "Synthesis and computer study of population dynamics controlled models using methods of numerical optimization, stochastization and machine learning," *Mathematics*, vol. 9, no. 24, p. 3303, 2021. DOI: `10.3390/math9243303`.

[13] A. V. Demidova, "Equations of population dynamics in the form of stochastic differential equations," *RUDN Journal of Mathematics, Information Sciences and Physics*, vol. 1, pp. 67–76, 2013, in Russian.

[14] A. V. Demidova, O. V. Druzhinina, and O. N. Masina, "Design and stability analysis of nondeterministic multidimensional populations dynamics models," *RUDN Journal of Mathematics, Information Sciences and Physics*, vol. 25, no. 4, pp. 363–372, 2017.

[15] I. N. Sinitsyn, O. V. Druzhinina, and O. N. Masina, "Analytical modeling and stability analysis of nonlinear broadband migration flows," *Nonlinear World*, vol. 16, no. 3, pp. 3–16, 2018, in Russian.

[16] Y. M. Svirezhev and D. O. Logofet, *Stability of biological communities.* Moscow: Nauka, 1978, in Russian.

[17] H. I. Freedman and B. Rai, "Can Mutualism alter Competitive Outcome: a Mathematical Analysis," *The Rocky Mountain Journal of Mathematics*, vol. 25, no. 1, pp. 217–230, 1995.

[18] Z. Lu and Y. Takeuchi, "Global asymptotic behavior in single-species discrete diffusion systems," *Journal of Mathematical Biology*, vol. 32, pp. 67–77, 1993. DOI: `10.1007/BF00160375`.

[19] X.-a. Zhang and L. Chen, "The linear and nonlinear diffusion of the competitive Lotka–Volterra model," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 66, no. 12, pp. 2767–2776, 2007. DOI: `10.1016/j.na.2006.04.006`.

[20] A. V. Demidova, O. V. Druzhinina, O. N. Masina, and E. D. Tarova, "Computer research of nonlinear stochastic models with migration flows," in *Proceedings of the Selected Papers of the 10th International Conference "Information and Telecommunication Technologies and Mathematical Modeling of High-Tech Systems" (ITTMM-2019). CEUR Workshop Proceedings*, vol. 2407, 2019, pp. 26–37.

[21] A. A. Petrov, O. V. Druzhinina, O. N. Masina, and I. I. Vasilyeva, "The construction and analysis of four-dimensional models of population dynamics taking into account migration flows," *Uchenye zapiski UlGU. Series: Mathematics and Information Technology*, no. 1, pp. 43–55, 2022, in Russian.

[22] I. I. Vasilyeva, "Computer modeling of the system of population dynamics taking into account the variation of migration parameters," *Uchenye zapiski UlGU. Series: Mathematics and Information Technology*, no. 2, pp. 21–30, 2022, in Russian.

[23] S. Cui and M. Bai, "Mathematical analysis of population migration and its effects to spread of epidemics," *Discrete and Continuous Dynamical Systems — B*, vol. 20, no. 9, pp. 2819–2858, 2015. DOI: `10.3934/dcdsb.2015.20.2819`.

[24] H. C. Tuckwell, "A study of some diffusion models of population growth," *Theoretical Population Biology*, vol. 5, no. 3, pp. 345–357, 1974. DOI: `10.1016/0040-5809(74)90057-4`.

[25] H. I. Freedman and P. Waltman, "Mathematical models of population interactions with dispersal. I: Stability of two habitats with and without a predator," *SIAM Journal on Applied Mathematics*, vol. 32, no. 3, pp. 631–648, 1977. DOI: `10.1137/0132052`.

[26] L. J. S. Allen, "Persistence and extinction in single-species reaction-diffusion models," *Bulletin of Mathematical Biology*, vol. 45, no. 2, pp. 209–227, 1983. DOI: `10.1016/S0092-8240(83)80052-4`.

[27] Y. Takeuchi, *Global dynamical properties of Lotka–Volterra systems*. Singapore: World Scientific, 1996.

[28] A. V. Demidova, O. V. Druzhinina, M. Jacimovic, O. N. Masina, and N. Mijajlovic, "Synthesis and analysis of multidimensional mathematical models of population dynamics," in *2018 10th International Congress on Ultra Modern Telecommunications and Control Systems and Workshops (ICUMT)*, vol. 2018-November, 2019, pp. 361–366. DOI: `10.1109/ICUMT.2018.8631252`.

[29] A. Demidova, O. Druzhinina, M. Jacimovic, O. Masina, N. Mijajlovic, N. Olenev, and A. Petrov, "The Generalized algorithms of global parametric optimization and stochastization for dynamical models of interconnected populations," in *Optimization and Applications*, Cham: Springer International Publishing, 2020, pp. 40–54. DOI: `10.1007/978-3-030-62867-3_4`.

[30] M. N. Gevorkyan, T. R. Velieva, A. V. Korolkova, D. S. Kulyabov, and L. A. Sevastyanov, "Stochastic Runge–Kutta software package for stochastic differential equations," in *Dependability Engineering and Complex Systems*, Cham: Springer International Publishing, 2016, pp. 169–179. DOI: `10.1007/978-3-319-39639-2_15`.

[31] M. Gevorkyan, A. Demidova, T. Velieva, A. Korolkova, D. Kulyabov, and L. Sevastyanov, "Implementing a method for stochastization of one-step processes in a computer algebra system," *Programming and Computer Software*, vol. 44, pp. 86–93, Mar. 2018. DOI: `10.1134/S0361768818020044`.

[32] A. Korolkova and D. Kulyabov, "One-step stochastization methods for open systems," *EPJ Web of Conferences*, vol. 226, p. 02 014, 2020. DOI: `10.1051/epjconf/202022602014`.

[33] A. P. Karpenko, *Modern search engine optimization algorithms. Algorithms inspired by nature [Sovremennyye algoritmy poiskovoy optimizatsii. Algoritmy vdokhnovlennyye prirodoy]*, 2nd ed. Moscow: N.E. Bauman MSTU, 2016, in Russian.

[34] D. Simon, *Algorithms for evolutionary optimization [Algoritmy evolyutsionnoy optimizatsii]*. Moscow: DMK Press, 2020, in Russian.

[35] A. A. Petrov, O. V. Druzhinina, and O. N. Masina, "Application of the computational intelligence method to modeling the dynamics of multidimensional population system," in *Data Science and Algorithms in Systems*, vol. 597, Cham: Springer International Publishing, 2023, pp. 565–575. DOI: `10.1007/978-3-031-21438-7_45`.

[36] R. Lamy, *Instant SymPy Starter*. Packt Publishing, 2013.

[37] T. E. Oliphant, "Python for scientific computing," *Computing in Science Engineering*, vol. 9, no. 3, pp. 10–20, 2007. DOI: `10.1109/MCSE.2007.58`.

[38] C. Fuhrer, J. Solem, and O. Verdier, *Scientific computing with Python. Second edition*. Packt Publishing, 2021.

[39] C. Hill, *Learning scientific programming with Python*, Second Edition. Cambridge: Cambridge University Press, 2020.

[40] N. Sillero, J. C. Campos, S. Arenas-Castro, and A. M. Barbosa, "A curated list of R packages for ecological niche modelling," *Ecological Modelling*, vol. 476, p. 110 242, 2023. DOI: `10.1016/j.ecolmodel.2022.110242`.

[41] O. V. Druzhinina, O. N. Masina, and E. D. Tarova, "Synthesis, computer research and stability analysis for the multidimensional models of the dynamics of the interconnected population," *Nonlinear World*, vol. 17, no. 2, pp. 48–58, 2019, in Russian. DOI: `10.18127/j20700970-201902-06`.

[42] I. I. Vasilyeva, O. V. Druzhinina, and O. N. Masina, "Design and research of population dynamic model "two competitors – two migration areas"," *Nonlinear World*, vol. 20, no. 4, pp. 60–68, 2022, in Russian. DOI: `10.18127/j20700970-202204-06`.

[43] C. W. Gardiner, *Handbook of stochastic methods: for Physics, Chemistry and the Natural Sciences*. Heidelberg: Springer, 1985.

[44] N. G. Van Kampen, *Stochastic processes in Physics and Chemistry*. Amsterdam: Elsevier, 1992.

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Построение, стохастизация и компьютерное исследование динамических популяционных моделей «два конкурента – два ареала миграции»

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Аннотация. При изучении детерминированных и стохастических популяционных моделей актуальными задачами являются формализация процессов с учётом новых эффектов, обусловленных взаимодействием видов, и развитие компьютерных методов исследования. Компьютерные методы исследования позволяют выполнить анализ траекторий многомерных популяционных систем. Мы рассматриваем модель «два конкурента – два ареала миграции», в которой учитывается внутривидовая и межвидовая конкуренция в двух популяциях, а также двунаправленная миграция обеих популяций. Для указанной модели мы учитываем вариативность параметров естественного воспроизводства видов. Предложено формализованное описание четырёхмерной модели «два конкурента – два ареала миграции» и её модификаций. С помощью реализации эволюционного алгоритма получен набор параметров, обеспечивающих сосуществование популяций в условиях конкуренции двух видов в основном ареале с учётом миграции этих видов. Исходя из полученного набора параметров найдено положительное состояние равновесия. Построены двумерные и трёхмерные проекции фазовых портретов. Осуществлена стохастизация модели «два конкурента – два ареала миграции» на основе метода построения самосогласованных одношаговых моделей. Для описания структуры модели использованы уравнения Фоккера–Планка. Выполнен переход к четырёхмерному стохастическому дифференциальному уравнению в форме Ланжевена. Для проведения численных экспериментов использован специализированный программный комплекс, предназначенный для построения и изучения стохастических моделей, а также разработана компьютерная программа на основе дифференциальной эволюции. Использованы алгоритмы генерирования траекторий винеровского процесса и многоточечных распределений и модификации метода Рунге–Кутты. В детерминированном и стохастическом случаях изучена динамика траекторий популяционно-миграционных систем. Проведён сравнительный анализ детерминированных и стохастических моделей. Результаты могут найти применение в задачах моделирования популяционных, экономических, демографических и химических систем.

Ключевые слова: модели динамики популяций, стохастические дифференциальные уравнения, одношаговые процессы, стохастизация, конкуренция, миграция, траекторная динамика, проекции фазовых портретов, компьютерное моделирование, программный комплекс