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Approximation of radial structure of unstable ion-sound modes in rotating magnetized plasma column by eikonal equation

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Abstract. The problem of the correct asymptotic construction of the radial structure of linearly unstable ion-sound electrostatic eigenmodes is studied. The eigenvalue problem with boundary conditions of the first and second kind (electrodynamic and hydrodynamic types) for the oscillations that propagate in a uniform cylindrical column of magnetized plasma along an axial homogeneous magnetic field is formulated. A method for constructing a discrete spectrum of small-scale unstable oscillations of the system based on the basic principles of geometric optics is proposed. The main idea of the method is an explicit idea of the type of boundary conditions — the conductivity and absorbing properties of the wall bounding the plasma cylinder. A dispersion relation for unstable small-scale modes destabilized due to the effects of differential rotation is derived from the Eikonal equation. For the correct construction instability growth rates spectra an universal recipe for the selection of radial wave numbers of small-scale eigenmodes in accordance with any of the types of boundary conditions is proposed.

Key words and phrases: plasma waves, plasma instabilities, geometrical optics, normal modes

1. Introduction

It is well known that rotation of the plasmas in magnetic field is a source of various instabilities [1–6]. The most common of them are of a convective nature and occur for non-axisymmetric flute-like perturbations with $m \neq 0$ and $k_{\parallel} = 0$ (m is the azimuthal wave-number of perturbations and $k_{\parallel}$ is the projection of the wave vector on the direction of magnetic field). Recently was shown that electrostatic axisymmetric ($m = 0$) perturbations with frequencies in the ion-sound region in uniform plasma column in homogeneous magnetic field are destabilized by rotation, if the generalized momentum of ions decreases with radius [7]. However, the question about radial structure of these unstable perturbations is still open.

In this paper we estimate a radial structure of untable eigenmodes in a differentially rotating magnetized plasma column. Solutions of wave equation for axisymmetric perturbations is obtained by geometrical optics approximation. The corresponding dispersion relation is derived from eikonal equation.

2. Wave equation

The general wave equation for axisymmetric perturbations of the electrostatic potential Φ in a differentially rotating uniform plasma column with cold ions and hot electrons immersed in the axial magnetic field takes the following form, [7]:

$$\frac{1}{r}\left(\frac{c_s^2}{\omega^2-\kappa(r)\omega_{Bi}^2}r\Phi'\right)'+\left(1-\frac{k_\|^2c_s^2}{\omega^2}\right)\Phi=0. \tag{1}$$

where c_s is the ion-sound speed, ω is the perturbation frequency, ω_{Bi} is the ion-cyclotron frequency and $k_\|$ is the wavelength along axial magnetic field. The variable

$$\kappa(r)=\left(1+2\frac{\Omega}{\omega_{Bi}}\right)\left(1+\frac{1}{r}\frac{(r^2\Omega)'}{\omega_{Bi}}\right) \tag{2}$$

defines the rotation profile of plasma Ω. Prime implies the radial derivative $d(...)/dr$. Without rotation, $\Omega=0$, Eq. (1) describes proradation of the ion-sound waves along cylindrical plasma waveguide.

Together with the boundary conditions (BC), Eq. (1) constitutes the eigenvalue problem. At the center of plasma column, $r=0$, the solution is required to be finite, $|\Phi(0)|<\infty$. On the inner wall the are two types of BCs are exist. For plasma column with ideally conducting wall at radius R, we require $\Phi(r=R)=0$, which provides zero tangential component of the perturbed electric field, $\tilde{E}_z=-k_\|\Phi$. If the inner wall does not absorb particles, we require $\Phi'(r=R)=0$ that corresponds the standard "no flux" BC in fluid dynamics, because displacement of ions is proportional to radial electric field $\tilde{E}_r=-\Phi'$ [7, 8].

3. Solution of eikonal equation for unstable perturbations

For an arbitrary profile of rotation $\Omega(r)$ the exact solution of the Eq. (1) does not exist. However, the studied problem can be solved asymptotically. At first, let us consider solution of the Eq. (1) in the following form

$$\Phi(r)=A\left(k,S(r)\right)\exp\left[ikS(r)\right], \tag{3}$$

where k is the radial wavenumber and $S(r)$ is the eikonal [9, 10]. In the lowest order by $1/k$ one can find the standard eikonal equation

$$(S')^2 = N^2. \tag{4}$$

Here N is the refractive index of he medium, which is in considered problem equals

$$k^2 N^2(r) = -\left(1 - \frac{k_\parallel^2 c_s^2}{\omega^2}\right)\left(\frac{\omega^2 - \kappa(r)\omega_{Bi}^2}{c_s^2}\right). \tag{5}$$

As easily seen, the instability occurs only when $\kappa(r) < 0$. For unstable solutions with growth rate $\gamma = -i\omega$ Eq. (5) gives

$$\hat{N}^2(r) = \left(1 + \frac{k_\parallel^2 c_s^2}{\gamma^2}\right)\left(\frac{\gamma^2 - |\kappa(r)|\omega_{Bi}^2}{k^2 c_s^2}\right) \geqslant 0 \tag{6}$$

and Eq. (3) have only one trivial solution

$$S(r) = \int_0^r \hat{N}^2(x)\, dx. \tag{7}$$

Finally, the desired solution of the wave equation (1) in the lowest order of geometrical optics approximation takes the form

$$\Phi(r,z,t) = \Phi_0 \exp\left[ik\int_0^r \hat{N}^2(x)\, dx\right] \exp\left[i(k_\parallel z - \omega t)\right], \tag{8}$$

where Φ_0 is the complex amplitude.

Thus, one can find that Eq. (8) could be used for construct eigenfunctions with discrete spectra of γ_n by discrete values of radial wavenumbers $n = \pi/kR$:

$$\Phi_n(r,z,t) = \Phi_0 \left\{\cos\left[k\int_0^r \hat{N}_n^2(x,\gamma_n)\, dx\right] + i\sin\left[k\int_0^r \hat{N}^2(x,\gamma_n)\, dx\right]\right\} \times \\ \times \exp\left[i(k_\parallel z - \omega t)\right]. \tag{9}$$

4. Conclusion

It is shown that a sufficiently complex wave equation (1) for the electrostatic ion-sound perturbation of magnetized plasma with an arbitrary profile of rotation can be solved analytically by the geometrical optics approximation. The radial structure of normal modes is constructed by Eq. (9) and their spectra are described by eikonal (7).

References

[1] A. B. Mikhailovskii, *Theory of plasma instabilities: Volume 1: Instabilities of a homogeneous plasma.* New York: Consultants Bureau, 1974.

[2] A. B. Mikhailovskii, *Theory of plasma instabilities: Volume 2: Instabilities of a homogeneous plasma.* New York: Consultants Bureau, 1974.

[3] E. P. Vedenov, R. Z. Velikhov, and R. Z. Zagdeev, “Stability of plasma,” *Sov. Phys. Usp.*, vol. 4, pp. 332–369, 1961.

[4] A. V. Timofeev, “Oscillations of inhomogeneous flows of plasma and liquids,” *Sov. Phys. Usp.*, vol. 13, pp. 632–646, 1971.

[5] A. B. Mikhailovskii *et al.*, “High-frequency extensions of magnetorotational instability in astrophysical plasmas,” *Plasma Physics Reports*, vol. 34, no. 8, pp. 678–687, 2008.

[6] D. A. Shalybkov, “Hydrodynamic and hydromagnetic stability of the couette flow,” *Physics-Uspekhi*, vol. 52, no. 9, pp. 915–935, 2009.

[7] N. A. Marusov *et al.*, “Stability of electrostatic axisymmetric perturbations in rotating Hall plasmas,” in *Proc. 47th EPS Conference on Plasma Physics*, EPS 2021, Sitges, Spain, 2021, p. 16 190.

[8] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics. Course of Theoretical Physics. Vol. 6.* New York: Pergamon Press, 1987.

[9] S. Borowitz, *Fundamentals of quantum mechanics: Particles, waves, and wave mechanics.* New York: Pergamon Press, 1967.

[10] A. V. Timofeev, “Geometrical optics and the diffraction phenomenon,” *Phys. Usp.*, vol. 48, pp. 609–613, 2005.

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Построение радиальной структуры неустойчивых ионно-звуковых колебаний во вращающейся замагниченной плазме при помощи уравнения эйконала

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Аннотация. Рассмотрена задача о корректном асимптотическом построении радиальной структуры линейно неустойчивых собственных электростатических колебаний ионно-звукового типа, распространяющихся в однородном цилиндрическом столбе замагниченной плазмы вдоль осевого однородного магнитного поля. В цилиндрической области пространства координат сформулирована задача на собственные значения с краевыми условиями первого и второго рода (электродинамического и гидродинамического типа) для волнового уравнения ионно-звуковых колебаний. На основе базовых принципов геометрической оптики предложен метод построения дискретного спектра мелкомасштабных неустойчивых колебаний исследуемой системы, в основе которого лежит явное представление о типе краевых условий — проводимости и поглощающих свойствах стенки, ограничивающей плазменный цилиндр. При помощи уравнения эйконала получено дисперсионное соотношение для неустойчивых собственных мелкомасштабных мод, дестабилизированных за счёт эффектов дифференциального вращения — неоднородного по радиусу профиля угловой скорости ионов, вращающихся вокруг оси симметрии, вдоль которой направлен вектор индукции магнитного поля. Для корректного построения спектра дискретных инкрементов неустойчивых колебаний предложен универсальный рецепт подбора радиальных волновых чисел мелкомасштабных собственных мод в соответствии с каким-либо из типов краевых условий.

Ключевые слова: волны в плазме, неустойчивости плазмы, геометрическая оптика, собственные колебания