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# On a dispersion curve of a waveguide filled with inhomogeneous substance

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**Abstract.** The paper discusses the relationship between the modes traveling along the axis of the waveguide and the standing modes of a cylindrical resonator, and shows how this relationship can be explored using the Sage computer algebra system. In this paper, we study this connection and, on its basis, describe a new method for constructing the dispersion curve of a waveguide with an optically inhomogeneous filling. The aim of our work was to find out what computer algebra systems can give when calculating the points of the waveguide dispersion curve. Our method for constructing the dispersion curve of a waveguide with optically inhomogeneous filling differs from those proposed earlier in that it reduces this problem to calculating the eigenvalues of a self-adjoint matrix, i.e., a well-studied problem. The use of a self-adjoint matrix eliminates the occurrence of artifacts associated with the appearance of a small imaginary addition to the eigenvalues. We have composed a program in the Sage computer algebra system that implements this method for a rectangular waveguide with rectangular inserts and tested it on SLE modes. The obtained results showed that the program successfully copes with the calculation of the points of the dispersion curve corresponding to the hybrid modes of the waveguide, and the points found fit the analytical curve with graphical accuracy even when with a small number of basis elements taken into account.

**Key words and phrases:** waveguide, Maxwell's equations, normal modes, partial radiation conditions

## 1. Introduction

In classical electrodynamics, there are two related spectral problems — the problem of the normal modes of the waveguide and the problem of eigenmodes of the resonator. Recall their formulations.

Let $G$ be a regular domain in $\mathbb{R}^3$, below referred to as a resonator. A non-trivial field of the form

$$\vec{E} = \vec{E}(x, y, z)e^{i\omega t}, \quad \vec{H} = \vec{H}(x, y, z)e^{i\omega t},$$

satisfying the system of homogeneous Maxwell equations and boundary conditions

$$\vec{n} \times \vec{E} = 0, \quad \vec{n} \cdot \vec{H} = 0,$$

is called the eigenmode, and the corresponding value of the positive parameter $\omega$ is called the eigenfrequency, instead of which the wave number $k = \omega/c$ is usually used. To find the eigenfrequencies, it is necessary to solve the eigenvalue problem

$$\mathrm{rot}\vec{E} = -ik\mu\vec{H}, \quad \mathrm{rot}\vec{H} = ik\epsilon\vec{E}$$

with the boundary conditions

$$\vec{n} \times \vec{E} = 0, \quad \vec{n} \cdot \vec{H} = 0.$$

This problem, like the scalar one, is written as an eigenvalue problem for a completely continuous self-adjoint operator, and therefore it has a discrete spectrum, which can be found approximately using the Ritz method [1, p. 181]. To theoretically substantiate this statement, special Sobolev spaces are introduced, the embedding theorems for which, unfortunately, have so far been proved under the assumption of a smooth boundary [2, § 6.1].

Let $S$ be a regular domain in $\mathbb{R}^2$, let us call the cylinder $S \times \mathbb{R}$ a waveguide. Assume that the axis $Oz$ of the used Cartesian coordinate system is directed along the axis of the cylinder. A non-trivial field of the form

$$\vec{E} = \vec{E}(x, y)e^{i\omega t - i\gamma z}, \quad \vec{H} = \vec{H}(x, y)e^{i\omega t - i\gamma z},$$

satisfying the system of homogeneous Maxwell equations and boundary conditions

$$\vec{n} \times \vec{E} = 0, \quad \vec{n} \cdot \vec{H} = 0,$$

is called the normal waveguide mode [3]. In this case, the parameter $\beta = \gamma/c$ is called the phase constant. To find it at a fixed frequency, it is required to solve the eigenvalue problem

$$\overline{\mathrm{rot}}\vec{E} = -ik\mu\vec{H}, \quad \overline{\mathrm{rot}}\vec{H} = ik\epsilon\vec{E}$$

with the boundary conditions

$$\vec{n} \times \vec{E} = 0, \quad \vec{n} \cdot \vec{H} = 0.$$

Here, $\overline{\mathrm{rot}}$ means a differential operator in which differentiation with respect to $z$ is replaced with multiplication by $-i\gamma$. The points of the $k\gamma$ plane, at which this problem has a nontrivial solution, form a certain curve called the waveguide dispersion curve.

The spectral problem for waveguide modes does not belong to any studied type. In the case when the filling of the waveguide is homogeneous, the complete system of waveguide modes can be composed of two types of modes: transverse magnetic ($H_z = 0$) and transverse electric ($E_z = 0$) ones. The Borgnis theory of functions makes it possible to reduce the study of such modes to the study of the Laplace operator spectrum. In this case, the dispersion curve turns out to be the union of a countable number of hyperbolas [3, 4].

Traditionally, this problem is written as an eigenvalue problem with respect to three field components. The choice of the three field components from the six components of the vectors $\vec{E}$ and $\vec{H}$ can be done in different ways, which leads to different formulations of the problem. A. N. Bogolyubov and T. V. Edakina [5, 6] and Frank Schmidt [7, 8] used the components of vector $\vec{H}$, in the papers by E. Lezar and D. Davidson [9] vector $\vec{E}$ was used, A. L. Delitsyn [10–13] formulated the problem in terms of $H_x, H_y, E_z$. Normal modes of an axially symmetric waveguide with a dielectric core were considered by N. A. Novoselova, S. B. Raevsky and A. A. Titarenko [14], as well as by A. L. Delitsyn and S. I. Kruglov [15]. Potentials can be used instead of fields, for example, four scalar functions, as proposed in Refs. [16–18]. Finally, with considerable efforts, it is possible to reduce the spectral problem to the study of the spectrum of a self-adjoint quadratic pencil [19].

Traditionally, the spectral problem of the theory of waveguides was considered as an eigenvalue problem with respect to the parameter $\gamma$, and the wave number was considered given. This approach is justified, since the problem of waveguide diffraction considers the incidence of a monochromatic wave, which partially passes through and is partially reflected from the inhomogeneity; in this case, transmitted and reflected waves arise, travelling from the inhomogeneity, but having the same frequency as the incident wave. On the other hand, to construct a dispersion curve, it is quite unnecessary to to calculate its points at a sequence of frequency values. It is possible to search for its points at fixed values of $\gamma$. What is of importance here is only the convenience of solving the problem.

It is obvious from physical considerations that there should be a simple relationship between the modes traveling along the waveguide axis and the standing modes of the cylindrical resonator. In this paper, we investigate this relationship and, based on it, describe a new method for constructing the dispersion curve of a waveguide with an optically inhomogeneous filling.

## 2. Relation between travelling and standing modes

We managed to express the relation between the travelling and standing modes by two theorems.

**Theorem 1.** *If the waveguide $S \times \mathbb{R}$ has a normal mode*

$$\vec{E} = \vec{E}(x, y)e^{i\omega t - i\gamma z}, \quad \vec{H} = \vec{H}(x, y)e^{i\omega t - i\gamma z}$$

*at certain values of $\omega, \gamma$, then the resonator $S \times [0, \pi n/\gamma]$ has an eigenmode at the same $\omega, \gamma$.*

**Theorem 2.** *If the resonator $S \times [0, L]$ has an eigenmode*

$$\vec{E} = \vec{E}(x, y, z)e^{i\omega t}, \quad \vec{H} = \vec{H}(x, y, z)e^{i\omega t}$$

*whose components have continuous derivatives at* $z = 0, L$*, then it also has an eigenmode*

$$E_{x,y} = E_{x,y}(x, y) \sin \frac{\pi n z}{L} e^{i\omega t}, \quad E_z = E_z(x, y) \cos \frac{\pi n z}{L} e^{i\omega_s t},$$
$$H_{x,y} = H_{x,y}(x, y) \cos \frac{\pi n z}{L} e^{i\omega t}, \quad H_z = H_z(x, y) \sin \frac{\pi n z}{L} e^{i\omega t},$$

*for some natural value* $n$*, and the waveguide* $S \times \mathbb{R}$ *has a normal mode at the same frequency* $\omega$ *and* $\gamma = \pi n/L$*.*

Proven theorems allow us to reduce the solution of the spectral problem of the theory of waveguides to the solution of the spectral problem of the theory of cylindrical resonators.

## 3. Eigenmodes of resonators

Let $G$ be a resonator filled with a substance characterized, generally speaking, by variable $\epsilon$ and constant $\mu$. In this case, it is convenient to exclude $\vec{E}$ from the system of Maxwell equations and write down the system of second-order equations

$$\mathrm{rot} \frac{1}{\epsilon} \mathrm{rot} \vec{H} = k^2 \mu \vec{H}, \tag{1}$$

to which the boundary conditions should be added

$$\vec{H} \cdot \vec{n} = 0, \quad \mathrm{rot} \vec{H} \times \vec{n} = 0 \tag{2}$$

Every non-trivial solution $\vec{H}$ of the problem (1), (2) has an eigenmode

$$\vec{E} = \frac{1}{ik\epsilon} \mathrm{rot} H e^{i\omega t}, \quad \vec{H} = \vec{H} e^{i\omega t}$$

of the resonator $G$. Let us multiply (1) by the test vector $\vec{F}$, integrate over $G$ and apply the integration by parts:

$$\iiint\limits_G \mathrm{rot} \vec{F}^* \cdot \mathrm{rot} \vec{H} \frac{dxdydz}{\epsilon} - k^2 \mu \iiint\limits_G \vec{F}^* \cdot \vec{H} dxdydz = 0. \tag{3}$$

The closure of the set of vectors $\vec{F} \in C^1(\overline{G})$ satisfying the condition $\mathrm{div}\, \vec{F} = 0$ in $G$ and $\vec{F} \cdot n = 0$ on its boundary, in the norm generated by the scalar product

$$(\vec{F}, \vec{H}) = \iiint\limits_G \mathrm{rot} \vec{F}^* \cdot \mathrm{rot} \vec{H} \frac{dxdydz}{\epsilon} + \mu \iiint\limits_G \vec{F}^* \cdot \vec{H} dxdydz,$$

is a Hilbert space, which we will denote as $\mathfrak{H}(G)$. The generalized eigenmode of the resonator $G$ is the nonzero vector $\vec{H} \in \mathfrak{H}(G)$ that satisfies the identity (3) for any vector $\vec{F} \in \mathfrak{H}(G)$.

It follows from the embedding theorem [2] that there exists a completely continuous self-adjoint operator $\hat{A}$ such that

$$\mu \iiint\limits_G \vec{F}^* \cdot \vec{H} dxdydz = \iiint\limits_G \mathrm{rot}\vec{F}^* \cdot \mathrm{rot}(\hat{A}\vec{H}) \frac{dxdydz}{\epsilon}.$$

Therefore, the relation (3) can be written as $\vec{H} = k^2 \hat{A}\vec{H}$.

Hence, the eigenfrequencies of the resonator form an infinitely large sequence, and the eigenmodes corresponding to them form an orthonormal basis in the space $\mathfrak{H}(G)$.

The eigenvalues of a self-adjoint, completely continuous operator can be found using the Ritz method, it is only necessary to choose the basis of the space $\mathfrak{H}(G)$ in a proper way.

## 4. Dispersion curve of a waveguide

Let us consider the points of intersection of the waveguide dispersion curve $S \times \mathbb{R}$ with the straight line $\gamma = \mathrm{const}$. Each such point corresponds to the normal mode of the waveguide, and, by virtue of theorem 1, to the eigenmode of the resonator $S \times [0, \pi/\gamma]$. The eigenfrequencies of this resonator form an infinite monotonic sequence $\omega_1, \omega_2, \dots$. By virtue of theorem 2, these points correspond to modes from which one can construct normal waveguide modes. From here a theorem immediately follows.

**Theorem 3.** *The points of intersection of the waveguide dispersion curve $S \times \mathbb{R}$ with the straight line $\gamma = const$ form a countable set of points, whose set of abscissas coincides with the set of the resonator eigenfrequencies $S \times [0, \pi/\gamma]$.*

Thus, in order to construct a dispersion curve, it is necessary to solve the self-adjoint problem of natural vibrations of a cylindrical resonator $G$ using the Ritz method. The choice of a basis is complicated by the fact that the elements of the space $\mathfrak{H}(G)$ must satisfy the condition $\mathrm{div}\,\vec{F} = 0$.

To construct such a basis, we took the eigenfunctions $\phi_n$ of the Dirichlet problem on the section of a cylinder

$$\Delta_2 \phi + \alpha^2 \phi = 0, \quad \phi|_{\partial S} = 0$$

and eigenfunctions $\psi_n$ of the Neumann problem on the section of a cylinder

$$\Delta_2 \psi + \beta^2 \psi = 0, \quad \left.\frac{\partial \psi}{\partial n}\right|_{\partial S} = 0.$$

We will take orthonormal systems $\phi_n$ and $\psi_n$ with respect to $L^2(S)$. With their help, we construct the basis formed by TM fields

$$\vec{H} = \begin{pmatrix} \partial_y \phi_n(x,y) \\ -\partial_x \phi_n(x,y) \\ 0 \end{pmatrix} \cos s\gamma z, \quad n, m \in \mathbb{N},$$

and another basis for the TE fields

$$\vec{H} = \begin{pmatrix} \gamma \partial_x \psi_n(x,y) \cos s\gamma z \\ \gamma \partial_y \psi_n(x,y) \cos s\gamma z \\ \beta_n^2 \psi_n(x,y) \sin s\gamma z \end{pmatrix}, \quad n \in \mathbb{N},\, m \in \mathbb{Z},\, m \geqslant 0,$$

where $\gamma = \pi/L$, and $s$ takes integer values, but for our case when $\epsilon$ is independent of $z$, it is sufficient to take $s = 1$.

Truncating the infinite system to the first $N$ basis functions, we reduce the problem of finding the wave numbers (3) to an algebraic eigenvalue problem

$$\hat{A}H = k^2 \hat{B}H. \tag{4}$$

If we agree to write the first $N$ TM waves, and then the first $N$ TE waves, then the matrices of this system will have a block form

$$\hat{A} = \begin{pmatrix} A_{11}, & A_{12} \\ A_{21}, & A_{22} \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} B_{11}, & 0 \\ 0, & B_{22} \end{pmatrix}.$$

We composed the matrix elements as integrals over a cylinder and took those integrals that could be calculated explicitly for $\epsilon(x,y)$. The matrix $\hat{B}$ turned out to be diagonal, and

$$B_{11} = \operatorname{diag}\left(\frac{\alpha_n^2 \mu}{2}\right), \quad B_{22} = \operatorname{diag}\left(\frac{\beta_n^2 \mu}{2}\left(\gamma^2 + \beta_n^2\right)\right).$$

Therefore, the generalized eigenvalue problem (4) is reduced to the standard eigenvalue problem:

$$\hat{D}\vec{H} = k^2 \vec{H}.$$

The elements of the symmetric matrix $\hat{D}$ are defined as double integrals:

$$d_{n,m} = \frac{\gamma^2}{\alpha_n \alpha_m} \iint\limits_S (\nabla\phi_n \cdot \nabla\phi_m) \frac{dxdy}{\epsilon\mu} + \alpha_n \alpha_m \iint\limits_S \phi_n \phi_m \frac{dxdy}{\epsilon\mu},$$
$$n, m = 1, \dots, N_1,$$

$$d_{n,m} = \frac{\gamma}{\alpha_n \beta_m} \sqrt{\beta_m^2 + \gamma^2} \iint\limits_S \frac{\partial \psi_m \phi_n}{\partial xy} \frac{dxdy}{\epsilon\mu},$$
$$n = 1, \dots, N_1,\; m = N_1 + 1, \dots N_1 + N_2,$$

$$d_{n,m} = \frac{1}{\beta_n \beta_m} \sqrt{\beta_m^2 + \gamma^2} \sqrt{\beta_n^2 + \gamma^2} \iint\limits_S \nabla\psi_n \cdot \nabla\psi_m \frac{dxdy}{\epsilon\mu},$$
$$n, m = N_1 + 1, \dots, N_1 + N_2.$$

# 5. Calculation of the points of the waveguide dispersion curve in the Sage system

As is usually the case when applying the Ritz method, the matrix elements are integrals, which in the general case are calculated approximately. To avoid this problem in the first tests, we considered a special case when all integrals are calculated in elementary functions.

Consider a rectangular waveguide $S = L_x \times L_y$ with uniform filling $\epsilon_0 = 1$ and $\mu = 1$ (see figure 1). Inside it we place a rectangular insert $S_1$ with constant filling $\epsilon_1$ and $\mu = 1$. In this case, $\phi_n$ and $\psi_n$ are products of sines and cosines, and the matrix elements are integrals of such products over rectangular regions.

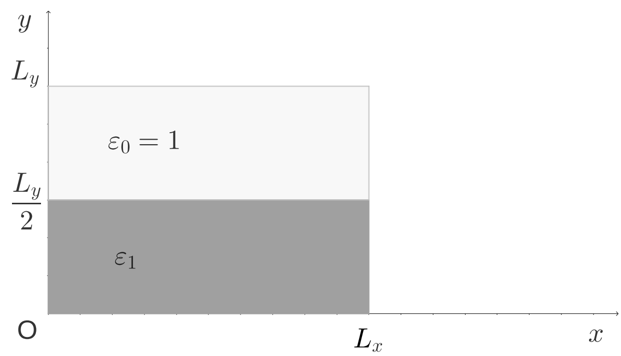


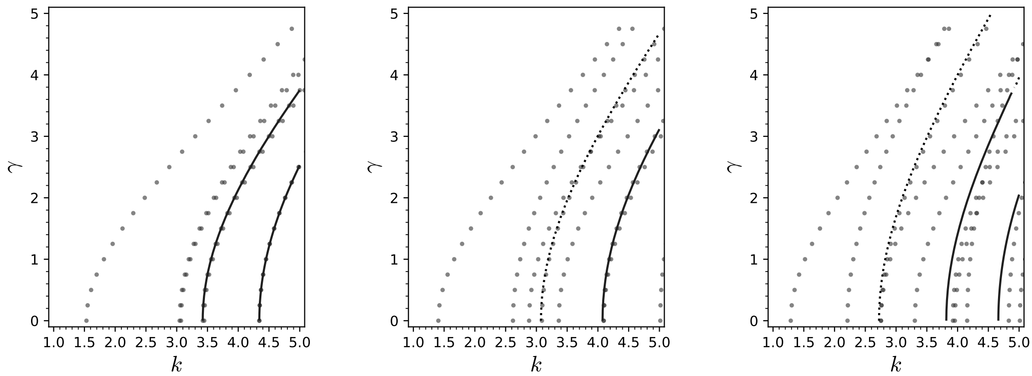

Figure 1. Waveguide cross section

We compiled a program (`https://github.com/malykhmd`) in the Sage computer algebra system that calculates these integrals in symbolic form, composes the $\hat{D}$ matrix for any given numerical value of $\gamma$, and calculates its eigenvalues using the standard `linalg` library function.

To test this program, we considered a waveguide in which the insert occupies the bottom half (see figure 1). In this case, two families of normal modes can be written analytically: SLE and SLH modes [4]. This example is interesting because these modes are hybrid and cannot be found by methods that ignore this effect.

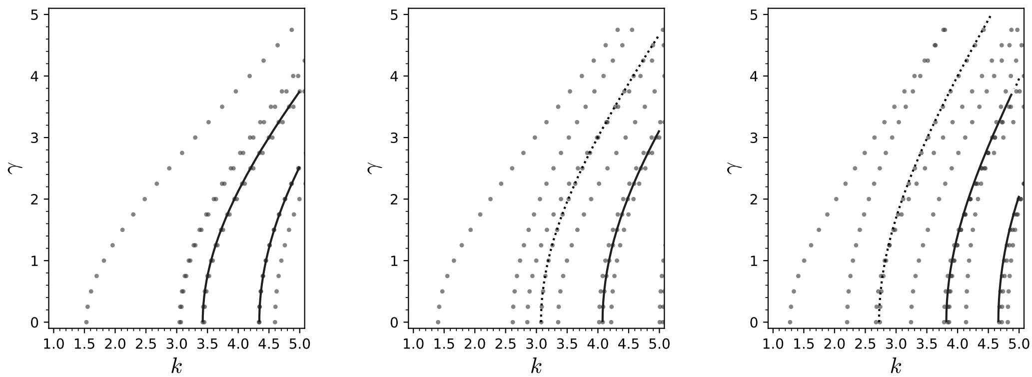
Figures 2, 3, and 4 demonstrate the results of analytical and numerical calculations performed using our program. It is clearly seen that for lower modes, the points found by our program fit the analytical curve with graphical accuracy even when a very small number of basis elements are taken into account (three for each direction). This is not hindered by the proximity of the neighboring arc due to the close values of $\epsilon_0$ and $\epsilon_1$. The accuracy of the calculations decreases with the growth of $\epsilon_1$, but when using 10 modes the calculated points fit the analytical curve even at $\epsilon_1 = 2$ (see figure 4).

The obtained results showed that the program successfully copes with the calculation of the points of the dispersion curve corresponding to the hybrid modes of the waveguide, and the points found fit the analytical curve with graphical accuracy even when with a small number of basis elements taken into account.



(a) $\epsilon_1 = 1.1$ (b) $\epsilon_1 = 1.5$ (c) $\epsilon_1 = 2$

Figure 2. Dispersion curve of a waveguide with an insert at $L_x = 1$, $L_y = 2$. The dots indicate the points of the dispersion curve found numerically (3 modes were taken in each of the directions), the solid lines are arcs of the dispersion curve corresponding to the SLE modes

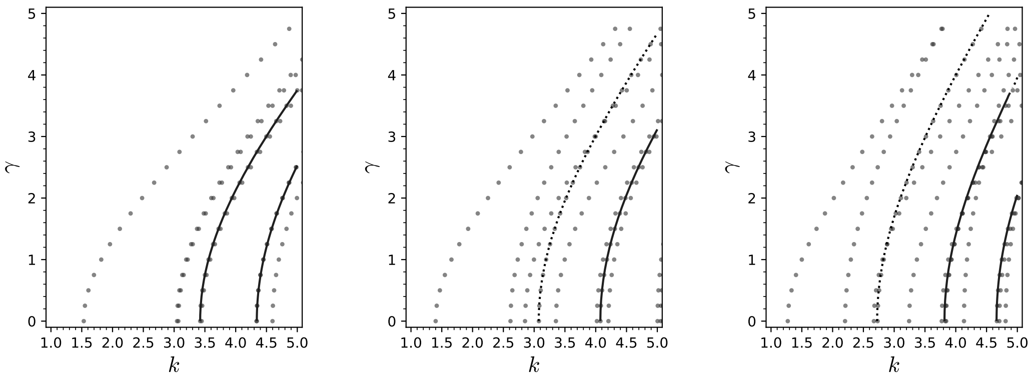


(a) $\epsilon_1 = 1.1$ (b) $\epsilon_1 = 1.5$ (c) $\epsilon_1 = 2$

Figure 3. Dispersion curve of a waveguide with an insert at $L_x = 1$, $L_y = 2$. The dots indicate the points of the dispersion curve found numerically (6 modes are taken in each of the directions), the solid lines are arcs of the dispersion curve corresponding to the SLE modes

## 6. Conclusion

Our approach to constructing the dispersion curve of a waveguide with an optically inhomogeneous filling differs from those proposed earlier in that it reduces the problem to calculating eigenvalues of a self-adjoint matrix, i.e., a well-studied problem. The use of a self-adjoint matrix eliminates the occurrence of artifacts associated with the appearance of a small imaginary addition to the eigenvalues.




(a) $\epsilon_1 = 1.1$ (b) $\epsilon_1 = 1.5$ (c) $\epsilon_1 = 2$

Figure 4. Dispersion curve of a waveguide with an insert at $L_x = 1$, $L_y = 2$. The dots indicate the points of the dispersion curve found numerically (10 modes were taken in each of the directions), thesolid lines are arcs of the dispersion curve corresponding to the SLE modes

We implemented this approach for an example of a rectangular waveguide with rectangular inserts in the Sage computer algebra system and tested it using SLE modes. At the same time, it was shown that our program perfectly copes with calculating the points of the dispersion curve corresponding to the hybrid modes of the waveguide.

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## References


[1] M. M. Karliner, *Microwave electrodynamics: Lecture course [Elektrodinamika SVCH: Kurs lektsiy].* Novosibirsk: Novosibirsk state iniversity, 2006, In Russian.

[2] G. Duvaut and J. L. Lions, *Les inéquations en mécanique et en physique.* Paris: Dunod, 1972.

[3] I. E. Mogilevsky and A. G. Sveshnikov, *Mathematical problems of diffraction theory [Matematicheskiye zadachi teorii difraktsii].* Moscow: MSU, 2010, In Russian.

[4] W. C. Chew, *Lectures on theory of microwave and optical waveguides.* 2012. DOI: `10.48550/arXiv.2107.09672`.

[5] A. N. Bogolyubov and T. V. Edakina, “Application of variation-difference methods for calculation of dielectric waveguides [Primeneniye variatsionno-raznostnykh metodov dlya rascheta dielektricheskikh volnovodov],” *Bulletin of Moscow University. Series 3: Physics. Astronomy*, vol. 32, no. 2, pp. 6–14, 1991, In Russian.

[6] A. N. Bogolyubov and T. V. Edakina, “Calculation of dielectric waveguides with complex cross-section shape by variation-difference method [Raschet dielektricheskikh volnovodov so slozhnoy formoy poperechnogo secheniya variatsionno-raznostnym metodom],” *Bulletin of Moscow University. Series 3: Physics. Astronomy*, vol. 34, no. 3, pp. 72–74, 1992, In Russian.

[7] P. Deuflhard, F. Schmidt, T. Friese, and L. Zschiedrich, “Adaptive Multigrid Methods for the Vectorial Maxwell Eigenvalue Problem for Optical Waveguide Design,” in *Mathematics — Key Technology for the Future*, W. Jäger and H. J. Krebs, Eds., Berlin–Heidelberg: Springer, 2011, pp. 279–292. DOI: 10.1007/978-3-642-55753-8_23.

[8] F. Schmidt, S. Burger, J. Pomplun, and L. Zschiedrich, “Advanced FEM analysis of optical waveguides: algorithms and applications,” *Proc. SPIE*, vol. 6896, 2008. DOI: 10.1117/12.765720.

[9] E. Lezar and D. B. Davidson, “Electromagnetic waveguide analysis,” in *Automated solution of differential equations by the finite element method*, The FEniCS Project, 2011, pp. 629–643.

[10] A. N. Bogolyubov, A. L. Delitsyn, and A. G. Sveshnikov, “On the completeness of the set of eigen- and associated functions of a waveguide [O polnote sistemy sobstvennykh i prisoyedinennykh funktsiy volnovoda],” *Computational Mathematics and Mathematical Physics*, vol. 38, no. 11, pp. 1815–1823, 1998, In Russian.

[11] A. N. Bogolyubov, A. L. Delitsyn, and A. G. Sveshnikov, “On the problem of excitation of a waveguide filled with an inhomogeneous medium [O zadache vozbuzhdeniya volnovoda s neodnorodnym zapolneniyem],” *Computational Mathematics and Mathematical Physics*, vol. 39, no. 11, pp. 1794–1813, 1999, In Russian.

[12] A. L. Delitsyn, “An approach to the completeness of normal waves in a waveguide with magnetodielectric filling,” *Differential Equations*, vol. 36, no. 5, pp. 695–700, 2000. DOI: 10.1007/BF02754228.

[13] A. L. Delitsyn, “On the completeness of the system of eigenvectors of electromagnetic waveguides,” *Computational Mathematics and Mathematical Physics*, vol. 51, no. 10, pp. 1771–1776, 2011. DOI: 10.1134/S0965542511100058.

[14] N. A. Novoselova, S. B. Raevsky, and A. A. Titarenko, “Calculation of propagation characteristics of symmetrical waves of round waveguide with radial non-uniform dielectric filling [Raschet kharakteristik rasprostraneniya simmetrichnykh voln kruglogo volnovoda s radial’no-neodnorodnym dielektricheskim zapolneniyem],” *Proceedings of Nizhny Novgorod State Technical University named after R.E. Alekseev*, vol. 2, no. 81, pp. 30–38, 2010, In Russian.

[15] A. L. Delitsyn and S. I. Kruglov, "Mixed finite elements used to analyze the real and complex modes of cylindrical waveguides," *Moscow University Physics Bulletin*, vol. 66, no. 6, pp. 546–551, 2011. DOI: 10.3103/S0027134911060063.

[16] A. A. Tiutiunnik, D. V. Divakov, M. D. Malykh, and L. A. Sevast'yanov, "Symbolic-Numeric Implementation of the Four Potential Method for Calculating Normal Modes: An Example of Square Electromagnetic Waveguide with Rectangular Insert," *Lecture notes in computer science*, vol. 11661, pp. 412–429, 2019. DOI: 10.1007/978-3-030-26831-2_27.

[17] D. V. Divakov, M. D. Malykh, L. A. Sevast'yanov, and A. A. Tiutiunnik, "On the Calculation of Electromagnetic Fields in Closed Waveguides with Inhomogeneous Filling," *Lecture notes in computer science*, vol. 11189, pp. 458–465, 2019. DOI: 10.1007/978-3-030-10692-8_52.

[18] M. D. Malykh and L. A. Sevast'yanov, "On the representation of electromagnetic fields in discontinuously filled closed waveguides by means of continuous potentials," *Computational Mathematics and Mathematical Physics*, vol. 59, no. 2, pp. 330–342, 2019. DOI: 10.1134/S0965542519020118.

[19] O. K. Kroytor, M. D. Malykh, and L. A. Sevast'yanov, "On normal modes of a waveguide," *Computational Mathematics and Mathematical Physics*, vol. 62, no. 3, pp. 393–410, 2022. DOI: 10.1134/S0965542522030083.



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# О дисперсионной кривой волновода, заполненного неоднородным веществом

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**Аннотация.** В статье рассматривается связь между модами, бегущими вдоль оси волновода, и стоячими модами цилиндрического резонатора. Показывается, как данная связь может быть исследована с помощью системы компьютерной алгебры Sage. В работе мы исследуем эту связь и на её основе описываем новый метод построения дисперсионной кривой волновода с оптически неоднородным заполнением. Целью нашей работы было выяснить, что могут дать системы компьютерной алгебры при вычислении (точек) дисперсионной кривой волновода. Метод построения дисперсионной кривой волновода с оптически неоднородным заполнением, предложенный нами, отличается от предложенных ранее тем, что сводит эту задачу к вычислению собственных значений самосопряжённой матрицы, то есть к задаче, хорошо изученной. Использование самосопряжённой матрицы исключает возникновение артефактов, связанных с появлением малой мнимой добавки у собственных значений. Мы составили программу в системе компьютерной алгебры Sage, в которой реализован этот метод для волновода прямоугольного сечения с прямоугольными вставками, и протестировали её на SLE-модах. Полученные результаты показали, что программа успешно справляется с вычислением точек дисперсионной кривой, отвечающих гибридным модам волновода, и найденные точки с графической точностью ложатся на аналитическую кривую даже при небольшом числе учитываемых базисных элементов.

**Ключевые слова:** волновод, уравнения Максвелла, нормальные моды, парциальные условия излучения