Complex eigenvalues in Kuryshkin–Wodkiewicz quantum mechanics

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Abstract. One of the possible versions of quantum mechanics, known as Kuryshkin–Wodkiewicz quantum mechanics, is considered. In this version, the quantum distribution function is positive, but, as a retribution for this, the von Neumann quantization rule is replaced by a more complicated rule, in which an observed value A is associated with a pseudodifferential operator $\hat{O}(A)$. This version is an example of a dissipative quantum system and, therefore, it was expected that the eigenvalues of the Hamiltonian should have imaginary parts. However, the discrete spectrum of the Hamiltonian of a hydrogen-like atom in this theory turned out to be real-valued. In this paper, we propose the following explanation for this paradox. It is traditionally assumed that in some state ψ the quantity A is equal to λ if ψ is an eigenfunction of the operator $\hat{O}(A)$. In this case, the variance $\hat{O}((A-\lambda)^2)\psi$ is zero in the standard version of quantum mechanics, but nonzero in Kuryshkin's mechanics. Therefore, it is possible to consider such a range of values and states corresponding to them for which the variance $\hat{O}((A-\lambda)^2)$ is zero. The spectrum of the quadratic pencil $\hat{O}(A^2) - 2\hat{O}(A)\lambda + \lambda^2\hat{E}$ is studied by the methods of perturbation theory under the assumption of small variance $\hat{D}(A) = \hat{O}(A^2) - \hat{O}(A)^2$ of the observable A. It is shown that in the neighborhood of the real eigenvalue λ of the operator $\hat{O}(A)$, there are two eigenvalues of the operator pencil, which differ in the first order of perturbation theory by $\pm i\sqrt{\langle\hat{D}\rangle}$.

Key words and phrases: models of quantum measurements, perturbation of discrete spectrum, complex eigenvalues, operator pencils

1. Introduction

The Kuryshkin–Wodkiewicz quantum mechanics [1] is an example of a dissipative quantum system. The quantum part of the measuring device is the

'environment of an open quantum system'. In the process of quantum measurement, an open quantum system interacts with its 'environment'. We study the result of this interaction [2]–[12]. Therefore, wave vectors must have a finite lifetime, inversely proportional to the imaginary part of eigenvalues.

In this version of quantum mechanics, the von Neumann quantization rule was abandoned and observable quantities are assigned to pseudo-differential operators, not necessarily self-adjoint. Therefore, the appearance of the imaginary part of the eigenvalues is not surprising. However, our studies of hydrogen-like atoms have shown that the operator corresponding to the Hamiltonian is essentially self-adjoint, so its discrete spectrum turned out to be real [13], [14].

This is quite surprising, since the von Neumann rule can be derived from general considerations, if we assume that the relation between the quantities $A = g(B)$ is inherited by their operators $\hat{A} = g(\hat{B})$ [15, P. 36]. Violation of this property inevitably means that the Kuryshkin–Wodkiewicz theory must be interpreted within the framework of the measurement theory and imaginary eigenvalues must appear. In this paper, we point out a spectral problem that has properties that are correct from this point of view.

2.   Quantization in Kuryshkin–Wodkiewicz mechanics

Consider a Hamiltonian system with coordinates $q \in \mathbb{R}^n$, momenta $q \in \mathbb{R}^n$, and Hamiltonian H. We will assume that the Hamiltonian and all observables considered below belong to a commutative ring $\mathcal{M}$, for example, to the polynomial ring $\mathbb{R}[p, q]$ or the ring $C^\infty(\mathbb{R}^n)[p]$.

In classical statistical mechanics, the state of the system is described by the distribution function f, in quantum mechanics by the wave function $\psi \in L^2(\mathbb{R}^n)$. In statistical mechanics, the mean value of the observable quantity $A \in \mathcal{M}$ is given by

$$\langle A \rangle = \iint\limits_{\mathbb{R}^{2n}} A(p, q) f dp dq,$$

and in quantum mechanics by the expression

$$\langle A \rangle = \int\limits_{\mathbb{R}^n} \psi^*(q) \hat{A} \psi(q) dq,$$

where $\hat{A}$ is the operator corresponding to the observable A. In 1966, Cohen [16] proved that these two equalities for the mean cannot be combined in one theory, if it is assumed that the density takes strictly positive values, and the transition from mechanical quantities to operators is carried out according to the von Neumann rule.

However, if this rule of 'quantization' of mechanical quantities is abandoned, then it is possible to construct a version of quantum mechanics in which the average can be calculated by both formulas and the density takes positive values. Instead of the von Neumann rule, this theory uses a more complicated

mapping of the commutative ring $\mathcal{M}$ into the ring of linear operators on the space $L^2(\mathbb{R}^n)$: $\hat{O}: \mathcal{M} \to L(L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n))$.

This correspondence does not satisfy the Neumann rule, i.e., generally speaking, $\hat{O}(A) \neq A(\hat{p}, \hat{q})$, but it is linear, namely: for any $A, B \in \mathcal{M}$ and any $k \in \mathbb{C}$

1. $\hat{O}(A + B) = \hat{O}(A) + \hat{O}(B)$,
2. $\hat{O}(kA) = k\hat{O}(A)$,
3. $\hat{O}(0) = 0$,
4. $\hat{O}(1) = \hat{E}$.

In the early 1970s, V. V. Kuryshkin [1] not only proved the existence of such mappings, but also proposed an explicit construction for them. In doing so, it was necessary to extend the class of operators, in which the mapping $\hat{O}$ takes value, from the class of self-adjoint differential operators to a non-commutative ring of non-self-adjoint pseudo-differential operators. The resulting new version of quantum mechanics was called Kuryshkin–Wodkiewicz mechanics.

It turned out that 'perturbed operators' satisfy a certain condition for the proximity of the new quantization rule to the von Neumann rule:

$$\hat{O}(A) = A(\hat{p}, \hat{q}) + \hat{V},$$

where the addition of $\hat{V}$ to the standard quantization rule is an operator compact in the sense of Jorgens [17]. Therefore, the lower bounds of the essential spectra of the operators $\hat{O}(A)$ and $A(\hat{p}, \hat{q})$, as well as the points of the discrete spectra of these operators, may not coincide, but the structure of the spectrum is preserved: the points of the discrete spectrum lie below the continuous spectrum [18].

For what follows, the explicit form of the mapping $\hat{O}$ is not important. For hydrogen-like atoms, we explicitly computed $\hat{O}(p_i)$ and $\hat{O}(g)$ for any function g of coordinates q [14]. It turned out that in all these cases self-adjoint operators are obtained. This implies, in particular, that the spectrum of the operator $\hat{O}(H)$ consists of a continuous part, which coincides with the spectrum $H(\hat{p}, \hat{q})$, found in standard quantum mechanics, below the lower boundary of which lie the discrete spectrum points, which are slightly different from the points of the discrete spectrum of the operator $H(\hat{p}, \hat{q})$. However, all these points are real due to the self-adjointness of the operator $\hat{O}(H)$.

3.   Spectral problem for a quadratic pencil

Let $A \in \mathcal{M}$ be an arbitrary observable. For brevity, we take

$$\hat{O}(A) = \hat{A}, \quad \hat{O}(A^2) = \hat{A}^2 + \hat{D}.$$

If the von Neumann rule is not satisfied, then two eigenvalue problems arise here:

1) classic problem

$$\hat{O}(A - \lambda)\psi = 0$$

or

$$\hat{A}\psi = \lambda\psi;$$

2) eigenvalue problem for a quadratic operator pencil

$$\hat{O}((A - \lambda)^2)\psi = 0$$

or

$$(\hat{A} - \lambda\hat{E})^2\psi + \hat{D}\psi = 0.$$

In standard quantum mechanics, $\hat{D} = 0$ and these problems are indistinguishable. The meaning of the first one has been discussed many times, but the second problem has a clear meaning. Expression

$$(\psi, O((A - \lambda)^2)\psi) = \langle(A - \lambda)^2\rangle$$

is the mean square deviation of the observable value A from the value λ for the system in the ψ state. In mechanics with a positive distribution function, which is the Kuryshkin–Wodkiewicz mechanics, this value coincides with

$$\langle(A - \lambda)^2\rangle = \iint\limits_{\mathbb{R}^{2n}} (A(p, q) - \lambda)^2 f\, dp\, dq$$

and therefore is non-negative. The same is true in standard quantum mechanics, but for a different reason:

$$\langle(A - \lambda)^2\rangle = \int\limits_{\mathbb{R}} (\mu - \lambda)^2 d(\psi, \hat{\sigma}_\mu\psi) \geqslant 0.$$

If we assume that $\hat{D}$ is small, then the eigenvalues of these spectral problems are close to each other. Let us study this circumstance in more detail.

But first, we note that in [19] we were looking for the ψ states that provide a minimum to this expression for fixed values of the parameter λ, for which we took the eigenvalues of the operator $\hat{A}$. It turned out that the minimum values are nonzero, that is, there is some nonzero variance. However, the problem can be formulated differently: to find the values of λ and the states ψ, at which the mean square deviation of the observed value A from λ is minimal. On the eigenfunctions of the pencil $\hat{O}((A - \lambda)^2)$, this standard deviation is zero, therefore, on the pencil eigenfunctions, the mean square deviation of the observable A from the eigenvalue λ has a minimum, i.e., zero value. Thus, we can observe the value A in 'pure' states corresponding to the eigenfunctions of the operator $\hat{A}$, or in 'pure' states that provide zero root-mean-square deviation A from some value other than $\langle A\rangle$.

We have already used perturbation theory [19] to find states with minimal variance, but now we will apply it to finding eigenfunctions of a quadratic pencil.

4.   Spectrum of a quadratic pencil

Let us introduce a small parameter ε and consider the problem

$$(\hat{A} - \lambda\hat{E})^2\psi + \varepsilon\hat{D}\psi = 0. \tag{1}$$

Let λ_0 be a single eigenvalue of the operator $\hat{A}$, and ψ_0 be the normalized eigenfunction corresponding to it. Let us study the eigenvalues of a quadratic pencil lying in a small neighborhood of this eigenvalue.

If the space under consideration is finite-dimensional, then all eigenvalues are roots of the determinant

$$\det\!\big((\hat{A} - \lambda\hat{E})^2 + \varepsilon\hat{D}\big) = 0.$$

In a neighborhood of the point $(\lambda, \varepsilon) = (\lambda_0, 0)$ the determinant

$$\det\!\big(\hat{A} - \lambda\hat{E}\big)^2 = \det\!\big(\hat{A} - \lambda\hat{E}\big)^2$$

has a zero of multiplicity 2, so

$$\det\!\big((\hat{A} - \lambda\hat{E})^2 + \varepsilon\hat{D}\big) = a(\lambda_0 - \lambda)^2 + \varepsilon b + \dots.$$

As is known from the theory of uniformization of curves [20], the curve

$$a(\lambda_0 - \lambda)^2 + \varepsilon b + \cdots = 0$$

in the plane $\lambda\varepsilon$ has a node at the point $(\lambda_0, 0)$ through which two arcs of this curve pass, which can be represented as two Puiseux series:

$$\lambda - \lambda_0 = \pm c\sqrt{\varepsilon} + \dots.$$

Thus, in the vicinity of a single eigenvalue of the operator $\hat{A}$ there are two eigenvalues of the quadratic pencil:

$$\lambda = \lambda_0 \pm \lambda_1\sqrt{\varepsilon} + \dots. \tag{2}$$

This can be justified in the case of infinite-dimensional spaces, for completely continuous operators $\hat{A}$, $\hat{D}$ this can be done using the well-known results of the perturbation theory of multiple eigenvalues [21]. Of course, in quantum theory, the operator $\hat{A}$ is pseudo-differential, and the question requires additional study. For the time being, we assume without further justification that the formally developed perturbation theory can be justified for this class of operators as well.

To find the first coefficient in the expansion, as in regular perturbation theory, we multiply (1) by ψ_0:

$$(\psi_0, (\hat{A} - \lambda\hat{E})^2\psi) = -\varepsilon(\psi_0, \hat{D}\psi). \tag{3}$$

Since the operator A is self-adjoint, we have

$$(\psi_0, (\hat{A} - \lambda\hat{E})^2\psi) = ((\hat{A} - \lambda^*\hat{E})^2\psi_0, \psi) = (\lambda_0 - \lambda)^2(\psi_0, \psi) = \varepsilon\lambda_1^2(\psi_0, \psi).$$

Substituting this expression into (3) and reducing by ε, we get

$$\lambda_1^2(\psi_0, \psi) = -(\psi_0, \hat{D}\psi).$$

Hence, in the limit $\varepsilon \to 0$, we have $\lambda_1^2 = -(\psi_0, \hat{D}\psi_0)$.

Substituting this expression into (2) and setting $\varepsilon = 1$, we have: in the neighborhood of eigenvalue λ_0 of the operator $\hat{A}$ there are two eigenvalues of the quadratic pencil $\hat{O}((A - \lambda)^2)$, namely $\lambda = \lambda_0 \pm i\sqrt{(\psi_0, \hat{D}\psi_0)} + ...$ where $\hat{D} = \hat{O}(A^2) - \hat{A}^2$.

5.　Conclusion and discussion

Let us now discuss the physical meaning of the resulting splitting of the eigenvalue of the operator $\hat{O}(A)$. The standard deviation of the observed value A from the value λ for a system in the ψ state is given by

$$\left(\psi, \hat{O}((A - \lambda)^2)\psi\right) = \langle(A - \lambda)^2\rangle.$$

This expression is non-negative both in standard quantum mechanics and in Kuryshkin–Wodkiewicz mechanics. It reaches zero on the eigenvectors of the quadratic pencil $\hat{O}((A - \lambda)^2)$.

In standard quantum mechanics

$$\hat{O}((A - \lambda)^2) = (\hat{A} - \lambda)^2$$

and therefore the eigenvectors of the pencil coincide with the eigenvectors of the operator $\hat{A}$. Therefore, the minimum standard deviation will be on those values of λ that are eigenvalues of the operator $\hat{A}$. They are traditionally considered as observed values of A.

In the mechanics of Kuryshkin–Wodkiewicz

$$\hat{O}((A - \lambda)^2) = (\hat{A} - \lambda)^2 + \hat{D}$$

and, as we just found out, the minimum standard deviation will be at those values of λ that differ from the eigenvalues λ_n of the operator $\hat{A}$ by $\pm i\sqrt{(\psi_n, \hat{D}\psi_n)}$.

Thus, the observed values of A will slightly differ from the eigenvalues of the operator $\hat{A}$. If $\langle\hat{D}\rangle > 0$, then this difference will manifest itself in the appearance of an imaginary additive, as one would expect in a dissipative quantum system. From this, two conclusions can be drawn.

Firstly, the transition to the root-mean-square deviation makes it possible to remove the difficulty with the reality of the spectrum of self-adjoint operators and obtain the expected dissipation in the Kuryshkin–Wodkiewicz mechanics.

Secondly, one of the two eigenvalues into which the eigenvalue $\hat{A}$ splits has the sign of the imaginary part corresponding to dissipation, and the second inevitably has a sign indicating antidissipation. We have already encountered a similar circumstance in the development of perturbation theory in the mathematical theory of waveguides [22], [23]: the spectral parameter λ should be considered as a point on the Riemann surface, only one sheet of which is physical, to which attention has been first drawn by V.P. Shestopalov [24]. In the case of Kuryshkin–Wodkiewicz mechanics, the eigenvalues of the operator $\hat{O}(A)$ are branch points on the Riemann surface, one of whose sheets describes a dissipative quantum system.

Acknowledgments

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Комплексные собственные значения в квантовой механике Курышкина–Вудкевича

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Аннотация. Рассматривается одна из возможных версий квантовой механики, известная как квантовая механика Курышкина–Вудкевича. В этой версии существует положительная квантовая функция распределения, но, в расплату за это, правило квантования фон Неймана заменено более сложным правилом, при котором наблюдаемой величине A ставится в соответствие псевдодифференциальный оператор $\hat{O}(A)$. Эта версия представляет собой пример диссипативной квантовой системы и поэтому ожидалось, что собственные значения гамильтониана должны иметь мнимые части. Однако точечный спектр гамильтониана водородоподобного атома в этой теории оказался вещественным. В настоящей статье мы предлагаем следующее объяснение этого парадокса. Традиционно принимают, что в некотором состоянии ψ величина A равна λ, если ψ — собственная функция оператора $\hat{O}(A)$. При этом дисперсия $\hat{O}((A - \lambda)^2)\psi$ равна нулю в стандартной версии квантовой механике, но не равна нулю в механике Курышкина. Поэтому можно рассмотреть такой спектр значений и соответствующих им состояний, при которых дисперсия $\hat{O}((A - \lambda)^2)$ равна нулю. В статье исследован спектр квадратичного пучка $\hat{O}(A^2) - 2\hat{O}(A)\lambda + \lambda^2\hat{E}$ методами теории возмущений в предположении малости дисперсии $\hat{D}(A) = \hat{O}(A^2) - \hat{O}(A)^2$ наблюдаемой A. Показано, что в окрестности вещественного собственного значения λ оператора $\hat{O}(A)$, имеется два собственных значения операторного пучка, которые в первом порядке теории возмущений различаются на величину $\pm i\sqrt{\langle \hat{D} \rangle}$.

Ключевые слова: модели квантовых измерений, возмущение дискретного спектра, комплексные собственные значения, пучки операторов