Mathematical analysis of a Markovian multi-server feedback queue with a variant of multiple vacations, balking and reneging

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Abstract. In this paper, we analyze a multi-server queue with customers' impatience and Bernoulli feedback under a variant of multiple vacations. On arrival, a customer decides whether to join or balk the system, based on the observation of the system size as well as the status of the servers. It is supposed that customer impatience can arise both during busy and vacation period because of the long wait already experienced in the system. The latter can be retained via certain mechanism used by the system. The feedback occurs as returning a part of serviced customers to get a new service. The queue under consideration can be used to model the processes of information transmission in telecommunication networks. We develop the Chapman–Kolmogorov equations for the steady-state probabilities and solve the differential equations by using the probability generating function method. In addition, we obtain explicit expressions of some important system characteristics. Different queueing indices are derived such as the probabilities when the servers are in different states, the mean number of customers served per unit of time, and the average rates of balking and reneging.

Key words and phrases: Markovian multi-server queue, probability generating function, impatient phenomena, server vacations, Bernoulli feedback

1. Introduction

Queueing models with server vacation have been efficiently studied by many researchers in the last decades and successfully applied in various practical problems such as telecommunication system design and control, manufacturing industries, and other related systems. There are two basic vacation queueing models namely, multiple vacation, and single vacation. In multiple vacation

queueing models, the server continues to take successive vacations until it finds at least one customer waiting in a queue at a vacation completion epoch [1], [2]. Nevertheless, in single vacation queueing models, the server precisely takes one vacation between two consecutive busy periods. These two types of vacation models were first introduced by Levy and Yechiali [3]. Eminent literature on the subject is found in [4]–[8] and others.

Over the past few years, queueing models with Bernoulli feedback have increasingly attracted the attention of many researchers [9]–[14]. Taking into account the feedback effect makes it possible to bring the considered models closer to a real situation, where the claims once serviced may require repeat service for different reasons. For example, in communication networks erroneously transmitted, a data is retransmitted.

In recent years, a growing body of literature has emerged on the analysis of queueing systems with impatient customers [15]. This is due to their potential applications in many related areas, see for instance [16], [17]. Balking is one form of impatience, which is the reluctance of a customer to join a queue upon arrival [18], [19]. The other forms are reneging, the reluctance to remain in line after joining and waiting, and jockeying between lines when each of a number of parallel lines has its own queue [20], [21]. When the impatience becomes sufficiently strong, the manager of the firm concerned has to take some measures to diminish the congestion to levels that customers can tolerate.

In most queueing situations, customers seem to get discouraged from receiving service when the server is absent and tend to leave the system without receiving service. This phenomenon is very precisely observed when the server is on vacation. This results in a potential loss of customers and customer goodwill for a service provider. For a comprehensive overview of the subject, authors may refer to [22]–[29]. Most of the literature mentioned here studies reneging during the vacation state of the server. However, in many real-life situations, the abandonment may occur even when the system is in the busy state. For instance, incoming customers can not have any information about the state of the server, or when they are not satisfied with the service time (in particular, when they find that the server takes too much time to serve the customers). This paper contributes in this sense. In fact, only a few research papers have been done treating this case [9]–[12], [30].

In this paper, we provide the analysis of a multi-server feedback queue with a variant multiple vacation policy, balking and server's states-dependent reneging. When all the customers present in the system have been served, the servers immediately leave for a vacation. If they return from a vacation to find an empty queue, they leave for another vacation; otherwise, the servers, synchronously, return to serve the queue. These latter are permitted to take a finite number, say K, of sequential vacations. It is assumed that an arriving customer who finds the system (all the servers) on vacation period (respectively, on busy period) activates an impatience timer T_{Vac} (respectively, T_{Busy}). If the customer's service has not been completed before the customer's impatience timer expires, the customer abandons the queue. The latter can be convinced to stay in the system (retained) using certain strategy. In addition, if the customer is unhappy with the service, he can rejoin the end of the queue for another one with some probability. That's what we call a feedback customer. To the best of the researchers' knowledge, the

model under consideration has so far not treated in the literature of queues. Moreover, our model can be considered as a generalized version of existing queueing model given by Yue et al. [27] and [20] equipped with many features and associated with many practical situations.

The rest of the paper is arranged as follows. In Section 2, we introduce the mathematical description of the model and we give a practical application. In Section 4, we develop the differential equations for the probability generating functions of the steady-state probabilities. In Section 5, we give the solution of the differential equations. In Section 6, we give the probabilities when the servers are in different states. Some essential system performance measures of this model are obtained in Section 7. Finally, we conclude the paper in Section 8.

2.   The mathematical description of the model

We consider a multi-server feedback queueing system with K-variant vacation, balking and server's states-dependent reneging. The following assumptions and notations are taken into account to structure the proposed queueing system:

1. The suggested queueing system consists of c servers. Customers arrive into the system according to a Poisson process with rate $\lambda > 0$, they are served according to First-Come-First-Served (FCFS) discipline. The service times are assumed to be exponentially distributed with rate μ.

2. A multiple synchronous vacation policy is considered; once all the customers present in the system are served, the servers, all together, leave for a vacation. At the end of the vacation period, if the queue is still empty, they immediately leave for another vacation; otherwise, they return to serve the queue. The servers are allowed to take all together K vacations sequentially. When the K consecutive vacations are complete, the servers switch to a busy period and, depending on the arrival of new customers, they stay idle or busy. The vacation period is assumed to be exponentially distributed with rate ϕ.

3. Whenever a customer arrives at the system and finds the servers on vacation period (resp. busy period), it activates an impatience timer T_{Vac} (resp. T_{Busy}), which is exponentially distributed with parameter ξ_0 (resp. ξ_1). If the customer's service has not been completed before the customer's timer expires, this later may leave the system. We suppose that the customers timers are independent and identically distributed random variables and independent of the number of waiting customers.

4. It is supposed that a system employs a certain mechanism in order to keep impatient customers in the system, that is, with some probability α', a customer may be retained in the system, and with a complementary probability α it may decide to leave to never return.

5. If, after completion of service, a customer is not happy with the quality of the service, he can return to the system with some probability β' for another service, or decide to leave the system with probability $\beta = 1 - \beta'$.

6. A customer who on arrival finds at least one customer (resp. c customers) in the system, when the servers are on vacation period (resp. busy

period) either decides to enter the queue with probability θ or balk with probability $\bar{\theta} = 1 - \theta$.

All random variables presented above are mutually independent of each other.

3.   Practical application of the model

The operation mode of a call center with vacation and impatience provides an initial motivation for our study; a central office is used for receiving or transmitting a large volume of enquiries. A private branch exchange (PBX) is a private telephone network used within a company or organizations that offers various features such as transfer calls, voicemail, call recording, interactive voice menus (IVR), and call queues. It helps in making an organization's communication simpler and more robust.

The incoming calls are routed to an available customer support manager drawn from the group of agents. Assume that the service facility consists in a group of c channels (servers) available to meet the demands of the requests. If an arriving call finds some servers free it immediately occupies the channel and leaves the system after service. However, the behavior of a call may vary depending on the waiting expectations provided by the call center and the personal preferences of each specific customer. Therefore, each call may decide either to balk or to wait for a while.

The servers commute between busy and vacation periods in groups. When there is no demands to be handled, the latter, all together, go synchronously on vacation and come back as one station to the busy period, once the idle period ends. If there are some waiting calls at the end of the vacation period, they will be immediately served. Alternatively, they quit for another vacation period.

The calls have no information on the queue length nor the state of the servers, then, an increase in the mean waiting time of a customer in the system can anticipate an increase in the average rate of reneging. Thus, to avoid losing potential customers, the system should employ some strategies by choosing the system parameter to further encourage customers to stay in the system. In the case that the service is not successful, the customer can repeat its request again and again until the service succeeds.

4.   Governing equations

At an arbitrary time, the system state is defined by a continuous time Markov chain $\{(L(t);\ J(t));\ t \geqslant 0\}$ on the state space $\Omega = \{(n;j) : n \geqslant 0;\ j = \overline{0, K}\}$, where $L(t)$ is the number of customers in the system and $J(t)$ is the state of the servers, i.e.,

$$J(t) = \begin{cases} j, & \text{if the servers are taking the } (j+1)^{\text{th}} \text{ vacation at time } t, \\ & j = \overline{0, K-1}, \\ K, & \text{if the servers are idle or busy at time } t. \end{cases}$$

Let $P_{n,j} = \lim_{t\to\infty} \mathbb{P}(L(t) = n;\ J(t) = j)$, $n \geqslant 0$; $j = \overline{0, K}$, denote the steady-state probabilities of the process $\{(L(t);\ J(t));\ t \geqslant 0\}$. The state-transition diagram is illustrated in Figure 1.
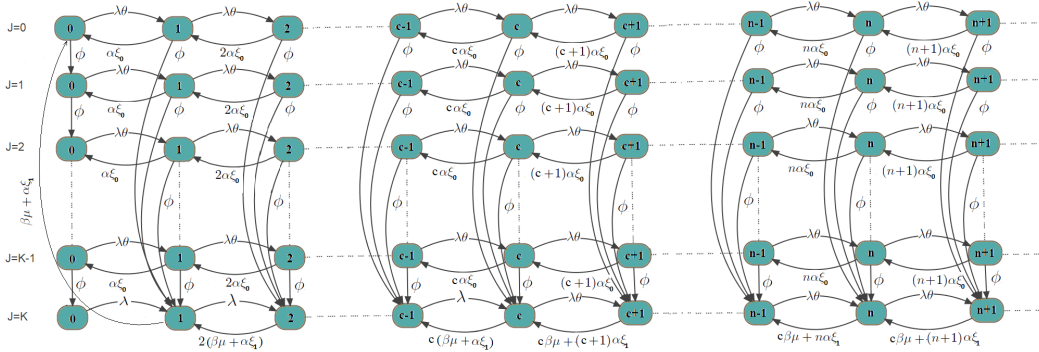

Figure 1. Transition plot

Using Chapman–Kolmogorov equations, we can formulate the balance equations for the suggested queueing model as:

$$(\lambda + \phi)P_{0,0} = \alpha\xi_0 P_{1,0} + (\beta\mu + \alpha\xi_1)P_{1,K}, \quad n = 0, \tag{1}$$

$$(\theta\lambda + \phi + \alpha\xi_0)P_{1,0} = \lambda P_{0,0} + 2\alpha\xi_0 P_{2,0}, \quad n = 1, \tag{2}$$

$$(\theta\lambda + \phi + n\alpha\xi_0)P_{n,0} = \theta\lambda P_{n-1,0} + (n+1)\alpha\xi_0 P_{n+1,0}, \quad n \geqslant 2, \tag{3}$$

$$(\lambda + \phi)P_{0,j} = \alpha\xi_0 P_{1,j} + \phi P_{0,j-1}, \quad j = \overline{1, K-1}, \quad n = 0, \tag{4}$$

$$(\theta\lambda + \phi + \alpha\xi_0)P_{1,j} = \lambda P_{0,j} + 2\alpha\xi_0 P_{2,j}, \quad j = \overline{1, K-1}, \quad n = 1, \tag{5}$$

$$(\theta\lambda + \phi + n\alpha\xi_0)P_{n,j} = \theta\lambda P_{n-1,j} + (n+1)\alpha\xi_0 P_{n+1,j}, \; j = \overline{1, K-1},\; n \geqslant 2, \tag{6}$$

$$\lambda P_{0,K} = \phi P_{0,K-1}, \quad n = 0, \tag{7}$$

$$(\lambda + \beta\mu + \alpha\xi_1)P_{1,K} = \lambda P_{0,K} + 2(\beta\mu + \alpha\xi_1)P_{2,K} + \phi\sum_{j=0}^{K-1} P_{1,j}, \quad n = 1, \tag{8}$$

$$(\lambda + n(\beta\mu + \alpha\xi_1))P_{n,K} =$$
$$= \lambda P_{n-1,K} + (n+1)(\beta\mu + \alpha\xi_1)P_{n+1,K} + \phi\sum_{j=0}^{K-1} P_{n,j}, \quad 2 \leqslant n \leqslant c-1, \tag{9}$$

$$(\theta\lambda + c\beta\mu + n\alpha\xi_1)P_{n,K} =$$
$$= \lambda P_{n-1,K} + (c\beta\mu + (n+1)\alpha\xi_1)P_{n+1,K} + \phi\sum_{j=0}^{K-1} P_{n,j}, \quad n = c, \tag{10}$$

$$(\theta\lambda + c\beta\mu + n\alpha\xi_1)P_{n,K} =$$
$$= \theta\lambda P_{n-1,K} + (c\beta\mu + (n+1)\alpha\xi_1)P_{n+1,K} + \phi\sum_{j=0}^{K-1}P_{n,j}, \quad n > c. \quad (11)$$

Consider the probability generating functions (PGFs) as:

$$G_j(z) = \sum_{n=0}^{\infty}z^n P_{n,j},$$

and define

$$G_j'(z) = \frac{d}{dz}G_j(z), \quad j = \overline{0, K}.$$

The normalizing condition is defined as

$$\sum_{n=0}^{\infty}\sum_{j=0}^{K}P_{n,j} = 1.$$

Multiplying Equation (3) by z^n, summing all possible values of n, and using Equations (1) and (2), we get

$$\alpha\xi_0(1-z)G_0'(z) - (\theta\lambda(1-z) + \phi)G_0(z) = -(\beta\mu + \alpha\xi_1)P_{1,K} + \overline{\theta}\lambda(1-z)P_{0,0}. \quad (12)$$

In the same manner, from Equations (4)–(6) and (7)–(11) respectively, we obtain

$$\alpha\xi_0(1-z)G_j'(z) - [\theta\lambda(1-z) + \phi]G_j(z) =$$
$$= \overline{\theta}\lambda(1-z)P_{0,j} - \phi P_{0,j-1}, \quad j = \overline{1, K-1}, \quad (13)$$

and

$$\alpha\xi_1 z(1-z)G_K'(z) - (1-z)(\theta\lambda z - c\beta\mu)G_K(z) =$$
$$= c\beta\mu(1-z)P_{0,K} + z(\beta\mu + \alpha\xi_1)P_{1,K} - \phi z\sum_{j=0}^{K-1}G_j(z)+$$
$$+ \phi z\sum_{j=0}^{K-2}P_{0,j} + \lambda\overline{\theta}z(1-z)\Gamma_1(z) - \beta\mu(1-z)\Gamma_2(z), \quad (14)$$

where

$$\Gamma_1(z) = \sum_{n=0}^{c-1}z^n P_{n,K} \text{ and } \Gamma_2(z) = \sum_{n=1}^{c-1}(n-c)z^n P_{n,K}.$$

5. Solution of the differential equations

For $z \neq 1$, Equation (12) can be written as follows:

$$G_0'(z) - \left[\frac{\theta\lambda}{\alpha\xi_0} + \frac{\phi}{\alpha\xi_0(1-z)}\right] G_0(z) = -\frac{\beta\mu + \alpha\xi_1}{\alpha\xi_0(1-z)}P_{1,K} + \frac{\overline{\theta}\lambda}{\alpha\xi_0}P_{0,0}. \quad (15)$$

Multiply both sides of Equation (15) by $e^{-\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{\frac{\phi}{\alpha\xi_0}}$, we get

$$\frac{d}{dz}\left(e^{-\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{\frac{\phi}{\alpha\xi_0}}G_0(z)\right) =$$

$$= e^{-\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{\frac{\phi}{\alpha\xi_0}}\left(\frac{\overline{\theta}\lambda}{\alpha\xi_0}P_{0,0} - \frac{(\beta\mu + \alpha\xi_1)}{\alpha\xi_0(1-z)}P_{1,K}\right).$$

Next, integrating the above equation from 0 to z, we obtain

$$G_0(z) = e^{\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}} \times$$

$$\times \left\{G_0(0) + \frac{\overline{\theta}\lambda}{\alpha\xi_0}P_{0,0}C_1(z) - \frac{\beta\mu + \alpha\xi_1}{\alpha\xi_0}P_{1,K}C_2(z)\right\}, \quad (16)$$

with

$$C_1(z) = \int_0^z e^{-\frac{\theta\lambda}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}}ds \text{ and } C_2(z) = \int_0^z e^{-\frac{\theta\lambda}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}-1}ds.$$

Since $G_0(1) = \sum_{n=0}^{\infty} P_{n,0} > 0$ and $z = 1$ is the root of denominator of the right hand side of Equation (16), we have that $z = 1$ must be the root of the numerator of the right hand side of Equation (16). So, we obtain

$$G_0(0) = \frac{(\beta\mu + \alpha\xi_1)P_{1,K}}{\alpha\xi_0}C_2(1) - \frac{\overline{\theta}\lambda P_{0,0}}{\alpha\xi_0}C_1(1), \quad (17)$$

where

$$C_1(1) = \int_0^1 e^{-\frac{\theta\lambda}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}}ds \text{ and } C_2(1) = \int_0^1 e^{-\frac{\theta\lambda}{\alpha\xi_0}s}(1-s)^{\frac{\phi}{\alpha\xi_0}-1}ds.$$

Noting $G_0(0) = P_{0,0}$. Then, Equation (17) implies

$$P_{1,K} = \frac{\alpha\xi_0}{(\beta\mu + \alpha\xi_1)C_2(1)}BP_{0,0} = \varpi_1 P_{0,0}, \quad (18)$$

with

$$B = 1 + \frac{\lambda}{\alpha\xi_0}\overline{\theta}C_1(1) \text{ and } \varpi_1 = \frac{\alpha\xi_0}{(\beta\mu + \alpha\xi_1)C_2(1)}B.$$

Substituting Equation (18) into Equation (16), we obtain

$$G_0(z) = e^{\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}}\left\{1 + \frac{\overline{\theta}\lambda}{\alpha\xi_0}C_1(z) - \frac{B}{C_2(1)}C_2(z)\right\}P_{0,0}. \qquad (19)$$

Next, Equation (13) can be written as

$$G_j'(z) - \left[\frac{\theta\lambda}{\alpha\xi_0} + \frac{\phi}{\alpha\xi_0(1-z)}\right]G_j(z) = \frac{\overline{\theta}\lambda}{\alpha\xi_0}P_{0,j} - \frac{\phi}{\alpha\xi_0(1-z)}P_{0,j-1}. \qquad (20)$$

Similarly, as for Equation (15), we multiply both sides of Equation (20) by $e^{-\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{\frac{\phi}{\alpha\xi_0}}$. Then, we find

$$G_j(z) = e^{\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}}\times$$
$$\times\left\{G_j(0) + \frac{\lambda\overline{\theta}}{\alpha\xi_0}C_1(z)P_{0,j} - \frac{\phi}{\alpha\xi_0}C_2(z)P_{0,j-1}\right\}, \quad j = \overline{1, K-1}. \qquad (21)$$

Since $G_j(1) = \sum\limits_{n=0}^{\infty} P_{n,j} > 0$ ($G_j(1) = P_{\bullet,j}$ represents the probability that the servers are taking the $(j+1)^{\text{th}}$ vacation) and $z = 1$ is the root of denominator of the right hand side of Equation (21), we have that $z = 1$ must be the root of the numerator of the right hand side of Equation (21). So, we obtain

$$G_j(0) = P_{0,j} = AP_{0,j-1}, \quad j = \overline{1, K-1}, \qquad (22)$$

where $A = \dfrac{\phi C_2(1)}{\alpha\xi_0 B}$. Using Equation (22) repeatedly, we get

$$P_{0,j} = A^j P_{0,0}, \quad j = \overline{1, K-1}. \qquad (23)$$

Now, by substituting Equation (23) into Equation (21), we find

$$G_j(z) = e^{\frac{\theta\lambda}{\alpha\xi_0}z}(1-z)^{-\frac{\phi}{\alpha\xi_0}}A^j\times$$
$$\times\left\{1 + \frac{\lambda\overline{\theta}}{\alpha\xi_0}C_1(z) - \frac{B}{C_2(1)}C_2(z)\right\}P_{0,0}, \quad j = \overline{1, K-1}. \qquad (24)$$

To find $P_{0,K}$; the probability that the servers are idle during the busy period, we use Equations (7) and (23). Thus

$$P_{0,K} = \varpi_0 P_{0,0}, \qquad (25)$$

where $\varpi_0 = \dfrac{\phi}{\lambda}A^{K-1}$.

Remark 1. It is easy to see that $0 < \phi C_2(1) < \alpha\xi_0$, and $\overline{\theta}\lambda C_1(1) > 0$. Thus, $0 < \phi C_2(1) < \alpha\xi_0 + \overline{\theta}\lambda C_1(1)$. Consequently, we have $0 < A < 1$.

Next, Equation (14) can be written as:

$$G'_K(z) - \left(\frac{\theta\lambda}{\alpha\xi_1} - \frac{c\beta\mu}{\alpha\xi_1 z} \right) G_K(z) = \frac{\beta\mu + \alpha\xi_1}{\alpha\xi_1(1-z)} P_{1,K} + \frac{c\beta\mu}{\alpha\xi_1 z} P_{0,K} + \frac{\lambda\bar{\theta}}{\alpha\xi_1} \Gamma_1(z) -$$

$$- \frac{\beta\mu}{\alpha\xi_1 z} \Gamma_2(z) + \frac{\phi}{\alpha\xi_1(1-z)} \left(\sum_{j=0}^{K-2} P_{0,j} - \sum_{j=0}^{K-1} G_j(z) \right). \quad (26)$$

In the same way, by multiplying Equation (13) by $\Upsilon(z) = e^{-\frac{\theta\lambda}{\alpha\xi_1}z} z^{\frac{c\beta\mu}{\alpha\xi_1}}$, we get

$$\frac{d}{dz} \left(\Upsilon(z) G_K(z) \right) = \Upsilon(z) \left\{ \frac{\beta\mu + \alpha\xi_1}{\alpha\xi_1(1-z)} P_{1,K} + \frac{c\beta\mu}{\alpha\xi_1 z} P_{0,K} + \frac{\lambda\bar{\theta}}{\alpha\xi_1} \Gamma_1(z) - \right.$$

$$\left. - \frac{\beta\mu}{\alpha\xi_1 z} \Gamma_2(z) + \frac{\phi}{\alpha\xi_1(1-z)} \left(\sum_{j=0}^{K-2} P_{0,j} - \sum_{j=0}^{K-1} G_j(z) \right) \right\}. \quad (27)$$

Then, integrating from 0 to z and using Equations (18) and (23)–(25), we obtain

$$G_K(z) = e^{\frac{\theta\lambda}{\alpha\xi_1}z} z^{-\frac{c\beta\mu}{\alpha\xi_1}} \left\{ \left((\beta\mu + \alpha\xi_1)\varpi_1 + \phi \left(\frac{1 - A^{K-1}}{1 - A} \right) \right) H_1(z) + \right.$$

$$+ \frac{c\beta\mu\phi}{\lambda} A^{K-1} H_2(z) - \phi \left(\frac{1 - A^K}{1 - A} \right) H_3(z) + \frac{1}{\alpha\xi_1} \times$$

$$\left. \times \left(\lambda\bar{\theta} \int_0^z s^{\frac{c\beta\mu}{\alpha\xi_1}} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} \Gamma_1(s) ds - \beta\mu \int_0^z s^{\frac{c\beta\mu}{\alpha\xi_1}-1} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} \Gamma_2(s) ds \right) \right\} P_{0,0}, \quad (28)$$

where

$$H_1(z) = \frac{1}{\alpha\xi_1} \int_0^z s^{\frac{c\beta\mu}{\alpha\xi_1}} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} (1-s)^{-1} ds,$$

$$H_2(z) = \frac{1}{\alpha\xi_1} \int_0^z s^{\frac{c\beta\mu}{\alpha\xi_1}-1} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} ds,$$

$$H_3(z) = \frac{1}{\alpha\xi_1} \int_0^z s^{\frac{c\beta\mu}{\alpha\xi_1}} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} \Psi(s)(1-s)^{-1} ds,$$

$$\Psi(s) = e^{\frac{\theta\lambda}{\alpha\xi_0}s} (1-s)^{-\frac{\phi}{\alpha\xi_0}} \left\{ 1 + \frac{\lambda\bar{\theta}}{\alpha\xi_0} C_1(s) - \frac{B}{C_2(1)} C_2(s) \right\}.$$

6. Evaluation of probabilities $P_{\bullet,K}$, $P_{\bullet,j}$ and $P_{0,0}$

From Equations (18) and (25), we have $P_{1,K} = \varpi_1 P_{0,0}$ and $P_{0,K} = \varpi_0 P_{0,0}$. Making use of Equations (4)–(6), we recursively get

$$\sum_{j=0}^{K-1} P_{n,j} = \delta_n P_{0,0},$$

where

$$\delta_n = \frac{1}{n\alpha\xi_0} \left\{ [\theta\lambda + \phi + (n-1)\alpha\xi_0]\delta_{n-1} - \theta\lambda\delta_{n-2} \right\}.$$

Similarly, from Equations (8), (9), we recursively obtain $P_{n,K} = \varpi_n P_{0,0}$, where

$$\varpi_n = \frac{1}{n(\beta\mu + \alpha\xi_1)} \left\{ [\lambda + (n-1)(\beta\mu + \alpha\xi_1)]\varpi_{n-1} - \lambda\varpi_{n-2} - \phi\delta_{n-1} \right\}.$$

Thus, Equation (28) can be written as

$$
\begin{aligned}
G_K(z) = \\
= e^{\frac{\theta\lambda}{\alpha\xi_1}z} z^{-\frac{c\beta\mu}{\alpha\xi_1}} &\left\{ \left[\frac{\alpha\xi_0 B}{C_2(1)} + \phi\left(\frac{1 - A^{K-1}}{1 - A}\right) \right] H_1(z) + \frac{c\beta\mu\phi}{\lambda} A^{K-1} H_2(z) - \right. \\
&\left. - \phi\left(\frac{1 - A^K}{1 - A}\right) H_3(z) + \lambda\bar{\theta}H_4(z) - \beta\mu H_5(z) \right\} P_{0,0}, \quad (29)
\end{aligned}
$$

with

$$H_4(z) = \frac{1}{\alpha\xi_1} \int_0^z s^{\frac{c\beta\mu}{\alpha\xi_1}} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} \Theta_1(s)ds, \quad H_5(z) = \frac{1}{\alpha\xi_1} \int_0^z s^{\frac{c\beta\mu}{\alpha\xi_1}-1} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} \Theta_2(s)ds,$$

$$\Theta_1(z) = \sum_{n=0}^{c-1} z^n \varpi_n, \quad \text{and} \quad \Theta_2(z) = \sum_{n=1}^{c-1} (n-c) z^n \varpi_n.$$

Thus, for $z = 1$ (noting that $G_K(1) = P_{\bullet,K}$ represents the probability that the servers are busy or idle), we get

$$G_K(1) = P_{\bullet,K} = \Phi(1)P_{0,0}, \quad (30)$$

where

$$
\begin{aligned}
\Phi(1) = e^{\frac{\theta\lambda}{\alpha\xi_1}} &\left\{ \left((\beta\mu + \alpha\xi_1)\varpi_1 + \phi\left(\frac{1 - A^{K-1}}{1 - A}\right) \right) H_1(1)+ + \right. \\
&\left. + \frac{c\beta\mu\phi}{\lambda} A^{K-1} H_2(1) - \phi\left(\frac{1 - A^K}{1 - A}\right) H_3(1) + \lambda\bar{\theta}H_4(1) - \beta\mu H_5(1) \right\},
\end{aligned}
$$

with

$$H_1(1) = \frac{1}{\alpha\xi_1} \int\limits_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} (1-s)^{-1} ds,$$

$$H_2(1) = \frac{1}{\alpha\xi_1} \int\limits_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}-1} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} ds,$$

$$H_3(1) = \frac{1}{\alpha\xi_1} \int\limits_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} \Psi(s)(1-s)^{-1} ds,$$

$$H_4(1) = \frac{1}{\alpha\xi_1} \int\limits_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} \Theta_1(s) ds,$$

$$H_5(1) = \frac{1}{\alpha\xi_1} \int\limits_0^1 s^{\frac{c\beta\mu}{\alpha\xi_1}-1} e^{-\frac{\theta\lambda}{\alpha\xi_1}s} \Theta_2(s) ds.$$

Now, from Equations (12) and (13), for $z = 1$, we have

$$P_{\bullet,j} = G_j(1) = A^{j-1} P_{0,0}, \quad j = \overline{0, K-1}. \tag{31}$$

By the definition of $P_{\bullet,j}$, using the normalizing condition, we get

$$\sum_{j=0}^{K} P_{\bullet,j} = 1.$$

Finally, from Equations (30) and (31), we get

$$P_{0,0} = \left(\frac{1 - A^K}{A(1-A)} + \Phi(1) \right)^{-1}. \tag{32}$$

7. Performance measures

The prime aim of determining probabilities in previous section is to formulate different metrics in order to examine the performance of the concerned system.

7.1. Mean system sizes

Systematic observations of the system state is very important to enhance the performance and to improve the decision-making.

Let L_j be the system size when the servers are in the state j $(j = \overline{0, K})$. Thus, $\mathbb{E}(L_j)$ is the mean system size when the servers are in the state j,

defined by

$$\mathbb{E}(L_j) = G'_j(1) = \sum_{n=1}^{\infty} n P_{n,j}, \quad j = \overline{0, K},$$

that is, for $j = \overline{0, K-1}$, $\mathbb{E}(L_j)$ represents the mean system size when the servers are taking the $(j+1)^{\text{th}}$ vacation, and $\mathbb{E}(L_K)$ represents the mean system size when the servers are busy. We first derive $\mathbb{E}(L_j)$ for $j = \overline{0, K-1}$.

From Equation (15), using the Hospital rule, we get

$$\mathbb{E}(L_0) = G'_0(1) =$$
$$= \lim_{z \to 1} \frac{-\theta\lambda G_0(z) + [\theta\lambda(1-z) + \phi]G'_0(z) - \lambda\overline{\theta}P_{0,0}}{-\alpha\xi_0} =$$
$$= \frac{\theta\lambda G_0(1) - \phi G'_0(1) + \overline{\theta}\lambda P_{0,0}}{\alpha\xi_0}.$$

Thus, we get

$$G'_0(1) = \frac{\theta\lambda G_0(1) + \lambda\overline{\theta}P_{0,0}}{\alpha\xi_0 + \phi}. \tag{33}$$

Similarly, from Equation (13), we find

$$(\alpha\xi_0 + \phi)G'_j(1) = \theta\lambda G_j(1) + \lambda\overline{\theta}P_{0,j}, \quad j = \overline{1, K-1}. \tag{34}$$

Then, from Equations (33) and (34), we have

$$\mathbb{E}(L_j) = G'_j(1) = \frac{\lambda[\theta G_j(1) + \overline{\theta}P_{0,j}]}{\alpha\xi_0 + \phi}, \quad j = \overline{0, K-1}. \tag{35}$$

By substituting Equation (31) and (35), we get

$$\mathbb{E}(L_j) = \frac{\lambda}{\alpha\xi_0 + \phi} \left[\frac{\theta + \overline{\theta}A}{A} \right] A^j P_{0,0}, \quad j = \overline{0, K-1}.$$

Thus, the mean system size when the servers are on vacation is obtained as

$$\mathbb{E}(L_V) = \sum_{j=0}^{K-1} \mathbb{E}(L_j) = \mathbb{E}(L_0) + \sum_{j=1}^{K-1} \mathbb{E}(L_j) =$$
$$= \frac{\lambda(\theta A^{-1} + \overline{\theta})}{(\alpha\xi_0 + \phi)} P_{0,0} + \frac{\lambda}{(\alpha\xi_0 + \phi)} \left[\frac{\theta + \overline{\theta}A}{A} \right] \sum_{j=1}^{K-1} A^j P_{0,0} =$$
$$= \left(\frac{\lambda(\theta + \overline{\theta}A)}{\alpha\xi_0 + \phi} \right) \left\{ \frac{2 - (A + A^{K-1})}{A(1-A)} \right\} P_{0,0}.$$

Next, from Equation (26) and by using the Hospital rule, we get

$$\mathbb{E}(L_K) = \lim_{z \to 1} G'_K(z) =$$

$$= \frac{1}{\alpha \xi_1} \left\{ (\theta \lambda - c\beta\mu)\Phi(1) + c\beta\mu \frac{\phi}{\lambda} A^{K-1} + \frac{\lambda\phi(\theta + \bar{\theta}A)}{\alpha \xi_0 + \phi} \left(\frac{1 - A^K}{A(1 - A)} \right) \right\} P_{0,0} +$$

$$+ \frac{1}{\alpha \xi_1} \left\{ \bar{\theta}\lambda\Theta_1(1) - \beta\mu\Theta_2(1) \right\} P_{0,0},$$

where $\Theta_1(1) = \sum\limits_{n=0}^{c-1} \varpi_n$ and $\Theta_2(1) = \sum\limits_{n=1}^{c-1} (n - c)\varpi_n$.

7.2.   Queueing model indices

The expressions for the mean queue length, the mean number of customers served and the average rates of impatient customers are established as follows:

— The mean size of the queue is calculated as

$$\mathbb{E}(L_q) = \sum_{j=0}^{K-1} \sum_{n=1}^{\infty} nP_{n,j} + \sum_{n=c}^{\infty} (n - c)P_{n,K} =$$

$$= \mathbb{E}(L) - c + \left\{ c \left[\frac{1 - A^K}{A(1 - A)} + \frac{\phi}{\lambda} A^{K-1} \right] - \Theta_2(1) \right\} P_{0,0}.$$

— The mean number of customers served per unit of time is given as

$$E_{\text{cs}} = \beta\mu \sum_{n=1}^{c-1} nP_{n,K} + c\beta\mu \sum_{n=c}^{\infty} P_{n,K} =$$

$$= \beta\mu \left\{ c + \left[\Theta_2(1) - c \left(\frac{\phi}{\lambda} A^{K-1} + \frac{1 - A^K}{A(1 - A)} \right) \right] P_{0,0} \right\}.$$

— The average rate of balking when the servers are in the state $j = \overline{0, K}$ is calculated as

$$B_{\text{r}} = \bar{\theta}\lambda \left(\sum_{j=0}^{K-1} \sum_{n=1}^{\infty} P_{n,j} + \sum_{n=c}^{\infty} P_{n,K} \right) =$$

$$= \bar{\theta}\lambda \left\{ 1 - \left[\frac{2 - A - A^{K-1} + (1 - A)\Theta_1(1)}{(1 - A)} \right] P_{0,0} \right\}.$$

— The average rate of abandonment of a customer due to reneging is as follows

$$R_{\text{ren}} = \sum_{j=0}^{K-1} \sum_{n=1}^{\infty} n\alpha\xi_0 P_{n,j} + \sum_{n=1}^{\infty} n\alpha\xi_1 P_{n,K} = \alpha\xi_0\mathbb{E}(L_V) + \alpha\xi_1\mathbb{E}(L_K).$$

8.   Conclusion

In this paper, we studied an $M/M/c$ feedback queue under synchronous K-variant vacations, balking, server's states-dependent reneging and retention of reneged customers. We developed the Chapman–Kolmogorov equations for the steady-state probabilities and solved the differential equations by using the probability generating function method. Based on these results, we obtained the probability generating function of the number of customers in the system when the system is on vacation period (resp. on busy period). In addition, we derived explicit expressions of some useful performance measures for the system. Furthermore, we presented closed-form expressions of some important other queueing indices such as the probabilities when the servers are in different states, the proportion of customers served per unit of time, and the average rates of balking and reneging.

It would be interesting to investigate a similar model with two-phase services and multiple vacation policy, server breakdown and repair, and customers' impatience. Further, one can evaluate the optimality of service and repair rates to minimize the waiting time of the customers in the system.

Acknowledgments

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Математический анализ марковской многолинейной системы массового обслуживания с обратной связью, прогулками приборов и нетерпеливыми заявками

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Аннотация. В работе исследуется система массового обслуживания с нетерпеливыми заявками, бернуллиевской обратной связью и прогулками приборов. В момент перед поступлением заявки в систему клиент, анализируя занятость системы и состояние приборов, принимает решение о принятии заявки или её уходе из системы. Предполагается, что нетерпение клиента может возникнуть как в период занятости, так и в период отдыха (прогулки) приборов из-за имевшихся ранее случаев длительного ожидания начала обслуживания в системе, информация о которых предоставляется с помощью определённого механизма. Обратная связь состоит в том, что часть ранее обслуженных клиентов может вернуться в систему для повторного обслуживания. Исследуемая система может применяться для анализа передачи данных в телекоммуникационных системах. Для стационарного распределения вероятностей записаны и решены с помощью производящих функций уравнения Колмогорова–Чепмена. Кроме того, получены аналитические выражения для ряда ключевых характеристик системы, например таких, как вероятности занятости или прогулки прибора, среднее число обслуженных заявок в единицу времени, средние интенсивности отказов от поступления и отказов от ожидания начала обслуживания.

Ключевые слова: марковская многолинейная система массового обслуживания, производящая функция, нетерпеливые заявки, прогулка прибора, обратная связь