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Richardson–Kalitkin method in abstract description

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An abstract description of the Richardson–Kalitkin method is given for obtaining a posteriori estimates for the proximity of the exact and found approximate solution of initial problems for ordinary differential equations (ODE). The problem \mathcal{P} is considered, the solution of which results in a real number u . To solve this problem, a numerical method is used, that is, the set $H \subset \mathbb{R}$ and the mapping $u_h : H \rightarrow \mathbb{R}$ are given, the values of which can be calculated constructively. It is assumed that 0 is a limit point of the set H and u_h can be expanded in a convergent series in powers of h : $u_h = u + c_1 h^k + \dots$. In this very general situation, the Richardson–Kalitkin method is formulated for obtaining estimates for u and c from two values of u_h . The question of using a larger number of u_h values to obtain such estimates is considered. Examples are given to illustrate the theory. It is shown that the Richardson–Kalitkin approach can be successfully applied to problems that are solved not only by the finite difference method.

Key words and phrases: finite difference method, ordinary differential equations, a posteriori errors

1. Introduction

A priori estimates for finding solutions to dynamical systems using the finite difference method predict an exponential growth of the error with increasing time [1]. Therefore, long-term computation requires such a small sampling step that cannot be accepted in practice. Nevertheless, calculations for long times are carried out and it is generally accepted that they reproduce not the coordinates themselves, but some average characteristics of the trajectories. In this case, a posteriori error estimates are used instead of huge a priori ones. As early as in the works of Richardson [2], for estimating the errors arising in the calculation of definite integrals by the method of finite differences, it was proposed to refine the grid, and in the works of Runge a similar technique



was applied to the study of ordinary differential equations. This approach was systematically developed in the works of N.N. Kalitkin and his disciples [3]–[7] as the Richardson method, although, given the role of Kalitkin in its development, it would be more correct to call it the Richardson–Kalitkin method.

The method itself is very general and universal, so we set out to present it in general form, divorcing it from the concrete implementation of the finite difference method. However, it soon became clear that this method could be extended to methods that are not finite difference methods, for example, the method of successive approximations, and even problems that are not related to differential equations.

In our opinion, this method is especially simply described for a class of problems in mechanics and mathematical physics, when it is necessary to calculate a significant number of auxiliary quantities, although only one value of some combination of them is interesting.

Example 1. On the segment $[0, T]$, we consider the initial problem

$$\frac{dx}{dt} = f(x, t), \quad x(0) = x_0,$$

it is required to find the value of x at the end of this segment, i.e., $x(T)$. To find this value numerically, we will have to calculate x approximately over the entire segment.

Example 2. On the segment $[0, T]$, we consider the dynamical system

$$\frac{dx}{dt} = f(x, y, t), \quad \frac{dy}{dt} = g(x, y, t),$$

with initial conditions $x(0) = x_0$, $y(0) = y_0$. It is required to find the value of the expression $x + y$ at the point $t = T$. To find this value numerically, we also have to calculate approximately x and y over the entire segment, then add the final values.

Example 3. The problem of many bodies is considered, say, the solar system, and it is required to find out whether the bodies scatter in 10 thousand years, or not. To solve it, it is enough to calculate the sum of the squares of the distances between the bodies and the center of mass of the system in 10 thousand years. At the same time, the coordinates and velocities of the bodies themselves are of no interest to anyone exactly 10 thousand years later.

Example 4. Let K be a unit circle on the plane. Find the first eigenvalue of the problem

$$\Delta v + \lambda v = 0, \quad v|_{\partial K} = 0.$$

Here, the eigenvalue λ_1 is to be found. We cannot find it numerically without finding the eigenfunction or roots of the determinant, i.e., other eigenvalues.

All these problems have one property in common: the result of the solution is a real number u . Various numerical methods are used to solve such

problems. To substantiate these methods, the errors that occur in intermediate calculations when calculating auxiliary parameters are estimated, and then they are summed up. The a priori error estimates obtained in this way turn out to be enormous. However, in many cases the real situation is much better than the forecasts obtained in this way. Using example 2, this can be explained as follows: errors usually made in the calculation of x and y have different signs and therefore their contributions to the expression $x + y$ are canceled. Having estimated the error in calculating $x + y$ as the sum of the modules of errors in determining x and y , we inevitably and significantly overestimate the error. It will not be superfluous to note that problems whose solution is just a real number are considered in the topology \mathbb{R} . This means that the numerical solution must be a number that is close to the exact solution in that topology. However, the topology of the space in which the auxiliary variables take values is not specified. Usually, numerical methods are constructed so that these auxiliary variables are found with greater accuracy with respect to some Euclidean norm. For example, to find $x + y$ at time T , you need to find an approximation to the pair of functions $x(t)$, $y(t)$ with respect to the norm

$$\sup_{0 \leq t \leq T} \sqrt{|x(t)|^2 + |y(t)|^2}.$$

In the situation under consideration, such requirements are unnecessarily stringent.

In this paper, we describe a method for obtaining estimates of errors made in solving problems of this class in general form based on the Richardson–Kalitkin method [3], [4], abstracting from the particular choice of numerical method. In our opinion, this approach makes it possible to clearly see the main ideas of the Kalitkin method, which usually turn out to be hidden behind the details of the numerical methods used. Half a century of using the Richardson–Kalitkin method in practice has shown that its correct application requires the calculation of not two, but a significantly larger number of approximate solutions to test the hypothesis of the dominance of the principal term in the error (see section 4 below). We will discuss one possible modification of the method for the simultaneous use of all of these solutions for evaluating solutions and errors.

2. Basic definitions

Let the problem \mathcal{P} be given, the solution of which is a real number u . We will not concretize this problem, let it only be known that this problem has a solution and, moreover, a unique one.

We are not going to concretize the numerical method for solving this problem. The use of any numerical method for solving it means replacing the problem \mathcal{P} with another problem \mathcal{P}_h , the result of which is the mapping $u_h : H \rightarrow \mathbb{R}$. The interpretation of the set H essentially depends on the numerical method used. In some cases this set is a segment $(0, \infty)$, and in other cases it consists of positive rational numbers. For example, for the finite difference method, this set is formed by the admissible step lengths. Below this does not matter, but it is important that the set H is a subset of the real axis and that 0 is a limit point for the set H .

By analogy with the usual conventions, let us accept the following

Definition 1. Let $u_h : H \rightarrow \mathbb{R}$ be a solution to the problem \mathcal{P}_h . If $\lim_{h \rightarrow 0} u_h = u$, then we say that the problem \mathcal{P}_h approximates the problem \mathcal{P} . If $u_h = u + \mathcal{O}(h^k)$, then we say that the order of approximation of problem \mathcal{P} by problem \mathcal{P}_h is k .

In the overwhelming majority of cases, the value h has the meaning of the discretization step of the original problem, and the order of k is known. Here are some examples.

Example 5. Let the problem \mathcal{P} consist in finding the value of the integral

$$u = \int_{x=0}^1 \frac{dx}{1+x^2}.$$

Its solution is the number $u = \pi/4$, which we do not know exactly. To calculate it, we cut the segment $[0, 1]$ into $N \in \mathbb{N}$ parts. Let us assume that H is formed by all possible inverse natural numbers. Let u_h map this set to \mathbb{R} , putting in correspondence to $h = \frac{1}{N}$ the number

$$\sum_{n=0}^N \frac{h}{1+(nh)^2}.$$

Then $u_h = \pi/4 + \mathcal{O}(h)$, i.e., the order of approximation obtained by the rectangle rule is 1.

Remark 1. It should be noted that the methods of the numerical calculation of some classes integrals are known when the error depends on the step value not linearly or quadratically, but exponentially [8], [9].

Example 6. Let us consider the problem from example 1. An explicit Euler scheme can be used to solve it. We cut the segment $[0, T]$ into $N \in \mathbb{N}$ parts and take $h = \frac{T}{N}$. Let us put this number in correspondence with the number $u_h = x_N$, which is calculated by the recurrent formulas

$$x_{n+1} = x_n + f(x_n, nh)h, \quad n = 0, \dots, N-1.$$

Moreover, it is possible to prove an a priori estimate for the error [1]:

$$|u + u - u_h| = |x(T) - x_N| \leq Ce^{aT}h,$$

where C, a are some constants depending only on f and the initial data x_0 , but not on h and T . This immediately implies that $u_h = x(T) + \mathcal{O}(h)$, i.e., the order of approximation by the problem obtained using the Euler scheme is 1.

Basically, the finite difference method will be applied further, but this is not at all necessary.

Example 7. To calculate $u = x(T)$ from example 1, one can use the sequential iteration method (Picard's method). Let $N \in \mathbb{N}$ be the number of iterations, let us take $h = 1/N$ and assign this number to the number u_h , which is calculated as follows. First, N functions are calculated by recurrent formulas

$$x_{n+1}(t) = \int_{\tau=0}^t f(x_n(\tau), \tau) d\tau, \quad n = 0, \dots, N-1,$$

and then $u_h = x_N(T)$. In this case $u_h \rightarrow u$ at $h \rightarrow 0$, i.e. the problem \mathcal{P}_h approximates the initial problem.

The problem \mathcal{P}_h should be simpler than the original one in the sense that it is possible to calculate the values of the mapping $u_h : H \rightarrow \mathbb{R}$ at all points of H . In practice, this possibility is limited both by an increase in the computational complexity when approaching $h = 0$, and by an increase in the role of the round-off error.

Definition 2. The value of the function u_h at any point of the set H will be called the approximate solution to the problem \mathcal{P} , and the modulus of the difference between this value and the solution to the problem \mathcal{P} is the error made when solving problems \mathcal{P} by method \mathcal{P}_h .

3. A posteriori error estimates

The Richardson–Kalitkin method can be separated from the finite difference method by adopting the following definition.

Definition 3. Let $u_h : H \rightarrow \mathbb{R}$ be a solution to the problem \mathcal{P}_h . If there exists a constant $c \neq 0$ such that $u_h = u + ch^k + \mathcal{O}(h^{k+1})$, then we will say that ch^k is the leading term of the approximation error for problem \mathcal{P} by problem \mathcal{P}_h .

Remark 2. In practice, it is usually assumed that the estimate $u_h = u + \mathcal{O}(h^k)$ implies the existence of a constant c such that $u_h = u + ch^k + \mathcal{O}(h^{k+1})$. Usually, this can be justified. But the definition 3 specifically states that $c \neq 0$. If $c = 0$, then one speaks of superconvergence of the method, because the order of approximation turns out to be greater than that predicted in theory. For difficulties in applying the Richardson–Kalitkin method in the case of superconvergence, see [10].

The essence of the Richardson–Kalitkin method is as follows. If we discard $\mathcal{O}(h^{k+1})$, then $u_h = u + ch^k$. We do not know the values of u and c , but we can calculate u_h for any value of h . Taking two such values, say h_1 and h_2 , we have a system of two linear equations

$$u_h(h_1) = u + ch_1^k, \quad u_h(h_2) = u + ch_2^k,$$

resolving which for u and c , we will find some estimates for these quantities. We are talking about estimates, not values, since they are obtained by discarding $\mathcal{O}(h^{k+1})$.

Definition 4. Let $u_h : H \rightarrow \mathbb{R}$ be a solution to the problem \mathcal{P}_h and there exists a constant $c \neq 0$ such that $u_h = u + ch^k + \mathcal{O}(h^{k+1})$. For any two $h_1, h_2 \in H$ the solution to the system

$$u_h(h_1) = u + ch_1^k, \quad u_h(h_2) = u + ch_2^k$$

with respect to u and c will be called the Richardson–Kalitkin estimate for the solution u to the problem \mathcal{P} and the coefficient c at the leading term of the approximation error. We will denote these estimates as $\tilde{u}(h_1, h_2)$ and $\tilde{c}(h_1, h_2)$, below we will often omit the indication of their dependence on h_1, h_2 , if this will not introduce ambiguity into presentation.

Example 8. Consider the initial problem

$$\dot{x} = -y, \quad \dot{y} = x, \quad x(0) = 1, \quad y(0) = 0,$$

and let it be required to find $u = x(1)$. We approximate it according to the explicit Euler scheme and calculate the approximate solution for $h_1 = 0.1$ and $h_2 = 0.01$ in Sage [11]:

$$u_h(h_1) = 0.5707904499, \quad u_h(h_2) = 0.543038634332351.$$

The solution of the system

$$u_h(h_1) = u + ch_1, \quad u_h(h_2) = u + ch_2$$

yields an estimate $\tilde{u} = 0.539955099269280$ for $u = \cos 1 = 0.540302305868140$, and for the coefficient of the leading term of the error $\tilde{c} = 0.308353506307201$.

The result looks very reasonable. With $h = 0.1$, we have an estimate for the error $\tilde{c}h = 0.0308$, while the error itself is 0.0304. With $h = 0.01$, we have an estimate for the error $\tilde{c}h = 0.00308$, while the error itself is 0.0027. The estimate for the solution differs from the solution by only $3.5 \cdot 10^{-4}$, which is an order of magnitude better than the result with the smallest step.

Richardson–Kalitkin estimates can also be performed in problems for the solution of which other numerical methods are used, while for specific methods such estimates themselves are well known, but under different names. For example, in this way the error is estimated when determining the eigenvalues by means of the finite element method (FEM) [12].

Example 9. Let it be required to find the smallest eigenvalue of the problem

$$\Delta v + \lambda v = 0, \quad v|_{\partial K} = 0$$

in the unit circle K . Then the answer is the number $u = \lambda_1$. Let us apply the FEM implementation in the system FreeFem++ [13]. The parameter h will be the value of $1/N$, where N is the number of points into which the circle is divided during triangulation. Then, when using linear elements, the smallest eigenvalue of the approximate problem is $u_h = u + ch^2 + \mathcal{O}(h^3)$.

Two-sided estimates for the error were obtained in the PhD thesis by Panin [14]. Let us take $h_1 = 1/20$ at random, and $h_2 = 1/100$, then

$$u_h(h_1) = 6.0173, \quad u_h(h_2) = 5.79292.$$

The solution of the system

$$u_h(h_1) = u + ch_1^2, \quad u_h(h_2) = u + ch_2^2$$

yields $\tilde{u} = 5.78357083333333$ against the exact value $u = j_1^2 = 5.783185962946785$, and for the coefficient of the leading term of the error we get $\tilde{c} = 93.4916666666667$.

The result looks very reasonable. For $h = 0.01$, we have an estimate for the error $\tilde{c}h^2 = 9.34 \cdot 10^{-3}$, while the error itself is $9.73 \cdot 10^{-3}$. The estimate for the solution differs from the solution by only $3.84 \cdot 10^{-4}$, which is an order of magnitude better than the result for the least h .

4. Justification of the Richardson–Kalitkin method

Justification of the Richardson–Kalitkin method consists of two parts: first, it is necessary to prove that the used numerical method satisfies the asymptotic formula

$$u_h = u + ch^k + \mathcal{O}(h^{k+1}).$$

Second, it is necessary to justify the possibility of omitting $\mathcal{O}(h^{k+1})$. The first step essentially depends on the numerical method used and its discussion is beyond the scope of this article. The second step, on the contrary, has nothing to do with the choice of a numerical method. Let us consider it in more detail. To discard the remainder $\mathcal{O}(h^{k+1})$, it must be substantially less than the principal term ch^k . For this purpose, first of all, c must be nonzero, which is indicated in definition 4. Further, the considered values of h should be sufficiently small. We have no a priori data to know in advance how small the chosen h should be. Finally, in practice, we cannot take h too small as well, when the round-off error becomes essential in the calculation of u_h .

In order to find a practically suitable interval of h values, N. N. Kalitkin and his disciples [5]–[7] have recommended to carry out calculations at least at 10 points rather than only two ones. Richardson’s method can be applied only for those h , for which the error versus the step plotted in the log-log scale using these points, lies on a straight line with the slope k known from the theory. If the steps are too large, this plot differs from the straight line due to the fact that the discarded $\mathcal{O}(h^{k+1})$ is still large, and if the steps are too small, the rounding error becomes essential. If the slope of the straight line differs from k , then the phenomenon of superconvergence takes place (see remark 2).

Example 10. Let us return to example 8 and find an approximate solution by the fourth-order Runge–Kutta method with 15 steps, starting from the step $dt = 0.1$, each time decreasing the step by two times. Taking the approximate value for $x(1) = \cos 1$, obtained at the smallest step, as exact, we can plot the dependence of the error $\Delta x = x_n - x_{15}$ on the step dt , see the figure 1. The plot clearly shows an inclined section with a slope of approximately 4, followed by a horizontal section, interpreted as a region where a round-off error prevents further refinement of the solution.

With this approach, several natural questions arise. First, the points never exactly fall on a straight line. Therefore, we need quantitative characteristics

for the site, which we will consider straight. How can we find them? Second, since approximate solutions were found not for two, but for many values of h , how can they be used to refine the solution? Third, the terms in power series do not have to form a monotonic sequence, therefore, for large h , the leading term can be significantly less than the next term. Can this possibility be taken into account explicitly?

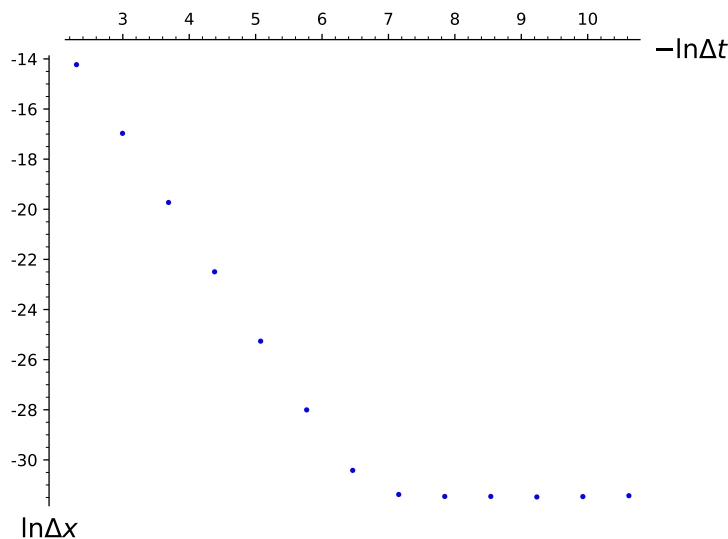


Figure 1. Dependence of error on step for example 10

5. Usage of several terms in the expansion of u_h in powers of h

The simplest answers to these questions can be found if we take into account the following terms in the expansion of u_h in powers of h . Suppose that u_h expands into a power series

$$u_h = u + c_1 h^k + c_2 h^{k+1} + \dots \quad (1)$$

If we have performed calculations for N different values for h , say, for $h = h_1, \dots, h_N$, then we can estimate the value of u and $N - 1$ coefficients, discarding all terms, starting with $c_N h^N$.

Definition 5. Let the solution $u_h : H \rightarrow \mathbb{R}$ to the problem \mathcal{P}_h be expanded in a power series (1), and let there be nonzero coefficients among c_1, \dots, c_{N-1} . For any N values $h_1, \dots, h_N \in H$ the solution to the system

$$u_h(h_n) = u + c_1 h_n^k + \dots + c_{N-1} h_n^{N+k-1}, \quad n = 1, 2, \dots, N \quad (2)$$

with respect to u and c_1, \dots, c_{N-1} will be called an estimate for the solution u to the problem \mathcal{P} and the first coefficients c over N approximate solutions. We will denote these estimates as \tilde{u} and $\tilde{c}_1, \dots, \tilde{c}_{N-1}$.

As a result of solving system (2) we have: i) the estimate \tilde{u} for the value of the exact solution, ii) the estimate $\tilde{c}_1 h^k$ for the error, suitable for sufficiently small h , and additional information about how small are those terms that are not taken into account in the Richardson–Kalitkin method.

Of course, as in the previous section, discarding terms, the order of which is equal to or greater than $N + k$ requires certain conditions to be met. However, these conditions are noticeably less restrictive. First, the simultaneous vanishing of several expansion coefficients seems incredible. Second, we can consider sufficiently large values of h for which the subsequent terms of the expansion are still noticeable.

6. Computer experiments

In our tests, we took $N = 4$ and $h_1 \in \mathbb{Q} \cap H$ at random, and the remaining h_2, h_3, h_4 were obtained by dividing h_1 by 2, 3 and 4. To avoid introducing additional rounding errors, system (2) is solved exactly over the field \mathbb{Q} .

Let us start with the simplest linear example.

Example 11. We will solve the problem from example 8 by the fourth-order Runge–Kutta method with a uniform step h . With step $h_1 = 0.1$, we get

$$u_h(0.1) = 0.540302967116884$$

against

$$\cos 1 = 0.540302305868140 \dots,$$

i.e., 6 correct decimal places. Calculating three more approximate solutions, we get the estimate for $u = \cos 1$ coinciding with the exact value up to 13 digits (the penultimate one). The estimate for the expansion coefficients (1) allows us to evaluate the error at $h = 0.1$ as

$$u_h - u = 0.007 \cdot 10^{-4} - 0.011 \cdot 10^{-5} + \dots = 6 \cdot 10^{-7},$$

as it should be. It is interesting to compare the interpolation polynomials obtained at the initial step $dt = 0.1$ and $dt = 0.01$: the estimate for $u = \cos 1$ coincides with the one obtained earlier up to the last digit, c_1 differs in the fifth digit, c_2 differs by an order of magnitude, and c_3 — by two orders of magnitude. We increased the number of bits allocated to a real number, and made sure that the noted effects are not related to round-off errors.

In the course of our experiments, we came across situations where the coefficients are monstrously overestimated.

Example 12. Consider the same system

$$\dot{x} = -y, \quad \dot{y} = x, \quad x(0) = 1, \quad y(0) = 0,$$

but let it be required to find $u = x(0.3)$. At the first step, $h_1 = 10^{-4}$, we got a huge estimate $\tilde{c} = 5 \cdot 10^{13}$, while the scatter of estimates is very high. However, the estimate for $\cos 0.3$ itself coincides with the exact value with a very high accuracy, and one can easily find such values for the initial step, at which the estimates for the coefficients look quite reasonable.

Application of the standard Richardson–Kalitkin method ($N = 2$) leads to even less pleasant results in this example. Take $h_1 = 0.1$ and $h_2 = 0.05$ and estimate u and c_1 using the Richardson–Kalitkin method. Then the estimate for the error $u_h - u$ will be $\tilde{c}_1 0.1^4 = 10^{-10}$, which is much less than the actual error $u_h(0.1) - \cos(0.1)$, equal to $2 \cdot 10^{-2}$.

The simplest explanation for these effects is that in the series

$$u_h = \cos 0.3 + c_1 h^4 + c_2 h^5 + \dots$$

the coefficient c_1 is very small, but the coefficient for some large power of h , on the contrary, is very large. Because of this, firstly, for small steps of the order of $h = 0.01$, we already have a value that coincides with the exact one, and, secondly, our estimates, which are based on the assumption of the possibility of discarding senior terms, do not work.

Now we proceed to a simplest nonlinear example.

Example 13. Let it be required to find $u = x(1)$ for solving the initial Volterra–Lotka problem

$$\dot{x} = (1 - y)x, \quad \dot{y} = -(1 - x)y, \quad x(0) = 0.5, \quad y(0) = 2$$

on the segment $0 < t < 1$. We will solve this problem according to the explicit Runge–Kutta scheme of the 4th order and estimate the solution with four steps, starting with $h_1 = 0.1$. For u , we obtain the estimate

$$u = 0.302408337777406,$$

and for the error

$$u_h(h) - u = -0.002 \cdot dt^4 + 0.00001 \cdot dt^5 + \dots$$

At the smallest step, we have an error of 10^{-9} , that is, we can rely on more than 9 decimal places. Starting with $h_1 = 0.01$ we get another estimate, in which \tilde{u} differs from the previously found value in the last two digits, and \tilde{c}_1 differs in the fourth digit.

7. Discussion of experimental results

The experiments performed, first of all, indicate that the proposed generalization of the Richardson–Kalitkin method allows, with a very modest number of steps, to obtain an estimate for the exact solution that coincides with it up to a round-off error. In this case, instead of 1 calculation, we perform 4 independent ones, which does not waste time at all, since the calculations are performed in parallel.

The larger the power, the greater the discrepancy in determining the coefficients for powers of h . It is not hard to explain this fact. All formulas are derived under an assumption typical of various kinds of mean-value theorems: for any $s > n$ there is a constant M_s such that

$$\left\| x - c_0 - \sum_{i=1}^s c_i h^{n+i-1} \right\| \leq M_s h^{n+s+1}.$$

When solving the interpolation problem, we solve the problem

$$c_0 + \sum_{i=1}^s c_i h_j^{n+i-1} = b_j + \xi_j h_j^{n+s+1},$$

where b_j are the values of x for $h = h_j$, and ξ_j are unknown quantities, about which we know that $|\xi_j| \leq M_s$.

Consider, for simplicity, $s = 2$

$$c_0 + c_1 h_1^n = b_1 + \xi_1 h_1^{n+1}, \quad c_0 + c_1 h_2^n = b_2 + \xi_1 h_2^{n+1}.$$

According to Cramer’s formulas

$$c_0 = \frac{b_2 h_1^n - b_1 h_2^n}{h_1^n - h_2^n} - (h_1 h_2)^n \frac{h_1 \xi_1 - h_2 \xi_2}{h_1^n - h_2^n},$$

and

$$c_1 = \frac{b_1 - b_2}{h_1^n - h_2^n} + \frac{h_1^{n+1} \xi_1 - h_2^{n+1} \xi_2}{h_1^n - h_2^n}.$$

For $h_1 = h, h_2 = h/2$, the error in c_0 will be of the order of $O(h^{n+1})$, and in c_1 — only of the order of $O(h)$. As s grows, the divergence of orders will become more and more noticeable.

Of course, the main problem is that we do not know neither s nor M_s . The example, in which superconvergence manifested itself, makes one think that there are cases when s cannot be taken as wanted. But in this case, the problem of applicability of the described method is reduced to the classical problem of the theory of power series: how many terms should be taken in the series in order to have a given accuracy? It is not difficult to answer it if the recurrent formulas for the coefficients are known, rather than the estimates for the coefficients of the power series, which become the worse the greater the power.

In theory, this circumstance is obviously a serious problem. However, in fact, all problematic cases immediately manifested themselves in the form of inadequately large coefficients. Thus, as a practical recipe, the generalization described seems to be quite useful.

8. Conclusion

We described the Richardson–Kalitkin method as a means for evaluating numerical methods for solving any problem \mathcal{P} , the result of which is a real number u . To specify a numerical method for solving the problem \mathcal{P} means to specify the set $H \subset \mathbb{R}$, for which 0 is a limit point, and the mapping $u_h : H \rightarrow \mathbb{R}$, the values of which can be calculated constructively. This method gives a solution to the problem \mathcal{P} , if $\lim_{h \rightarrow 0} u_h = u$.

If there exist $k \in \mathbb{N}$ and numbers c_1, \dots, c_N , among which there are nonzero numbers such that

$$u_h = u + ch^k + \dots c_N h^{k+N} + \mathcal{O}(h^{k+N+1}),$$

then from N values of the mapping u_h it is possible to estimate the exact solution of the original problem and the coefficients c_1, \dots, c_N , characterizing the error of the numerical method. The examples show that the higher the coefficient number, the worse these estimates are, but on the whole they characterize the numerical method quite accurately. The values of u_h are calculated independently, so the calculation of such problems can be naturally parallelized.

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Метод Ричардсона–Калиткина в абстрактном изложении

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Дано абстрактное описание метода Ричардсона–Калиткина для получения апостериорных оценок близости точного и найденного приближённого решения начальных задач для обыкновенных дифференциальных уравнений (ОДУ). Рассматривается задача \mathcal{P} , результатом решения которой является вещественное число u . Для решения этой задачи используется численный метод, то есть заданы множество $H \subset \mathbb{R}$ и отображение $u_h : H \rightarrow \mathbb{R}$, значения которого имеют возможность вычислять конструктивно. При этом предполагается, что 0 является предельной точкой множества H , u_h можно разложить в сходящийся ряд по степеням h : $u_h = u + c_1 h^k + \dots$. В этой весьма общей ситуации сформулирован метод Ричардсона–Калиткина получения оценок для u и c по двум значениям u_h . Рассмотрен вопрос об использовании большего числа значений u_h для получения такого рода оценок. Приведены примеры, иллюстрирующие теорию. Показано, что подход Ричардсона–Калиткина с успехом может быть применён к задачам, которые решаются не только методом конечных разностей.

Ключевые слова: метод конечных разностей, обыкновенные дифференциальные уравнения, апостериорные ошибки