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Applying Friedmann models to describe the evolution of the Universe based on data from the SAI Supernovae Catalog

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In the recent years thanks to the modern and sophisticated technologies the astronomers and astrophysicists were able to look deep into the Universe. This vast data poses some new problem to the cosmologists. One of the problems is to develop an adequate theory. Another one is to fit the theoretical results with the observational one. In this report within the scope of the isotropic and homogeneous Friedman–Lemaitre–Robertson–Walker (FLRW) cosmological model we study the evolution of the Universe filled with dust or cosmological constant. The reason to consider this model is the present universe surprisingly homogeneous and isotropic in large scale. We also compare our results with the data from the SAI Supernovae Catalog. Since the observational data are given in terms of Hubble constant (H) and redshift (z) we rewrite the corresponding equations as a functions of z. The task is to find the set of parameters for the mathematical model of an isotropic and homogeneous Universe that fits best with the astronomical data obtained from the study of supernovae: magnitude (m), redshift (z).

Key words and phrases: fitting, cosmology, Friedmann's Universe, data analysis

1. Introduction

Based on modern data, it is established that the universe is not stationary today, but it is expanding with acceleration [1]–[8]. This fact was established by studying large amounts of data on supernovae, including those that are remote in huge distances. The peculiarity of this work is that the objects contained in the database [9], [10] are located mainly at distances of non-cosmological scale, which may be reflected in the descriptive power of certain Friedmann models of the Universe.

2. Friedmann model

The equations of General relativity describing the evolution of the Universe are complex enough to solve them exactly. So Friedmann suggested that we instead accept two simple assumptions: (1) the universe looks exactly the same in all directions; (2) this condition holds true for all its points. Based on General relativity and these two simple assumptions, Friedmann showed [11] that the universe may not be stationary. This model was further independently developed by Lemaitre [12], Robertson [13]–[15] and Walker [16].

The equations describing the evolution of the Universe, and which we will solve, look like this:

$$\begin{cases} \dot{H} + H^2 = -\dfrac{4\pi G}{3}(\varepsilon + 3p), \\ \dot{a} = Ha, \\ \dot{\varepsilon} = -3H(\varepsilon + p). \end{cases} \tag{1}$$

Here: H — Hubble parameter, a — scale factor, ε — energy density, p — pressure. To solve this system, we need another condition — the connection between p and ε.

Let $p = f(\varepsilon)$. This relationship is called the equation of state. In our case (the dust Universe), this equation reduces to a trivial one: $p = 0$.

Let's go back to system (1), which is a system of differential equations with respect to time. First, we need to go from time to redshift, since the observational data contains this value.

The red-shift is defined as

$$z = \frac{\lambda_o - \lambda}{\lambda}, \tag{2}$$

here λ_o is the wavelength during detection, λ is the wavelength during emission.

$$1 + z = \frac{\lambda_o}{\lambda} = \frac{\nu_o}{\nu} = \frac{a(t_o)}{a(t)}, \tag{3}$$

$$\frac{dz}{dt} = -\frac{a(t_o)}{a^2(t)}\dot{a}(t) = -\frac{a(t_o)}{a(t)}\frac{\dot{a}(t)}{a(t)} = -(z+1)H. \tag{4}$$

Using equations (4), (3), and the second equation from the system (1), we obtain the first equation for the scale factor with respect to z:

$$\frac{da}{dz} = -\frac{a(t_o)}{(z+1)^2}. \tag{5}$$

Similarly for the equation on H:

$$\frac{dH}{dz}\frac{dz}{dt} + H^2 = -\frac{4\pi G}{3}(\varepsilon + 3p) => \frac{dH}{dz} = \frac{1}{1+z}\left[H + \frac{4\pi G}{3H}(\varepsilon + 3p)\right]. \quad (6)$$

The third equation from system (1) is converted to an equation with respect to a. Then the result (1) is rewritten as:

$$\begin{cases} \dfrac{dH}{dz} = \dfrac{1}{1+z}\left[H + \dfrac{4\pi G}{3H}(\varepsilon + 3p)\right], \\ \dfrac{da}{dz} = -\dfrac{a(t_o)}{(z+1)^2}, \\ \dfrac{d\varepsilon}{\varepsilon + p} = -3\dfrac{da}{a}. \end{cases} \quad (7)$$

Since the astronomical data contains a magnitude, it is necessary to associate the Hubble parameter with the magnitude. Let's do this with another equation — the equation for the distance that light travels:

$$\frac{dD}{dt} = c => \frac{dD}{dz}\frac{dz}{dt} = c => \frac{dD}{dz} = -\frac{c}{(1+z)H}, \quad (8)$$

and the equation for magnitude:

$$m = -2.5 \lg \frac{D_0^2}{D^2}, \quad (9)$$

The complete system of equations will then take the form:

$$\begin{cases} \dfrac{dH}{dz} = \dfrac{1}{1+z}\left[H + \dfrac{4\pi G}{3H}(\varepsilon + 3p)\right], \\ \dfrac{da}{dz} = -\dfrac{a(t_o)}{(z+1)^2}, \\ \dfrac{d\varepsilon}{\varepsilon + p} = -3\dfrac{da}{a}, \\ \dfrac{dD}{dz} = -\dfrac{c}{(1+z)H}, \\ m = -2.5 \lg \dfrac{D_0^2}{D^2}. \end{cases} \quad (10)$$

With the condition $p = 0$:

$$\varepsilon = \varepsilon_o(1+z)^3. \quad (11)$$

Substitute (11) and $p = 0$ in the first equation from (10), then the solution of this dif. equations with the initial condition $H(0) = H_0$:

$$H = \sqrt{\frac{8\pi G\varepsilon_0}{3}(1+z)^3 + \left(H_0^2 - \frac{8\pi G\varepsilon_0}{3}\right)(1+z)^2}. \tag{12}$$

Substitute (12) in the fourth equation from (10):

$$\alpha = H_0^2 - \frac{8\pi G\varepsilon_0}{3}, \quad \beta = \frac{8\pi G\varepsilon_0}{3}.$$

$$D = -c\int \frac{dz}{(1+z)^2[\alpha + \beta(z+1)]^{\frac{1}{2}}}. \tag{13}$$

Substituting $\beta x + \alpha = t^2$ and integrating with the condition $D(0) = 0$, we get:

$$D = -\frac{\beta}{\alpha^{\frac{3}{2}}} \ln \frac{\sqrt{\alpha} + \sqrt{\alpha + \beta(1+z)}}{\sqrt{1+z}(\sqrt{\alpha} + \sqrt{\alpha+\beta})} - \\ - \frac{\sqrt{\alpha+\beta}(1+z) - \sqrt{\alpha + \beta(1+z)}}{\alpha(1+z)}. \tag{14}$$

The results of the approximation are shown in the table 1.

Table 1

The result of the approximation in the case of $w = 0$ (dust)

RMSE	MSE	H_0, (km/s)/Mpc	Ω_m	D_0, m
1.4438	1.0478	68.0001	5.7764	9.7200×10^{20}

In the third equation from (10) we get rid of time and substitute $p = 0$, we get:

$$\varepsilon = \varepsilon_0 a^{-3}. \tag{15}$$

Now substitute H from the second equation from (1) and (15) to the first equation from (1):

$$\ddot{a} = -\frac{4\pi G}{3}\varepsilon_0 a^{-2}. \tag{16}$$

We also solve it numerically with initial conditions $a(0) = 1$, $\dot{a}(0) = H_0$.

3. LCDM-model

LCDM-model is short for Lambda-Cold Dark Matter, a modern cosmological model based on the assumption of isotropy and homogeneity of the Universe [17]. The space environment in this model consists of several components: dark energy (Λ-member), cold dark matter, ordinary matter, and radiation.

Table 2

LCMD-model parameters

H_0, (km/s)/Mpc	Ω_m	Ω_{rad}	ε_{crit}, kg/m^3	Ω_Λ
67.74	0.3082	0.0001	8.62×10^{-27}	0.6911

Model parameters [18] are shown in table 2.

The first and fourth equations from (10) will take the form:

$$\begin{cases} \dfrac{dH}{dz} = \dfrac{H}{1+z} + \dfrac{H_0^2}{H}\left(\Omega_{rad}(1+z)^3 + \dfrac{1}{2}\Omega_m(1+z)^2 - \Omega_\Lambda(1+z)^{-1}\right), \\ \dfrac{dD}{dz} = -\dfrac{c}{(1+z)H}. \end{cases} \tag{17}$$

And equation for the scale factor:

$$\ddot{a} = -H_0^2\left(\Omega_{rad}a^{-3} + \frac{1}{2}\Omega_m a^{-2} - \Omega_L a\right). \tag{18}$$

We also solve it numerically with initial conditions $\dot{a}(0) = H_0$, $a(0) = 1$. RMSE and MSE for the LCDM model are 1.4612 and 1.0579, respectively.

4. Comparison of models

Comparison of theoretical results and observations within the scope of LRS Bianchi type-I model was performed in [19]–[21].

An approximation of the m(z) curve using the parameters of the LCDM model and the Friedmann model is shown in figure 1.

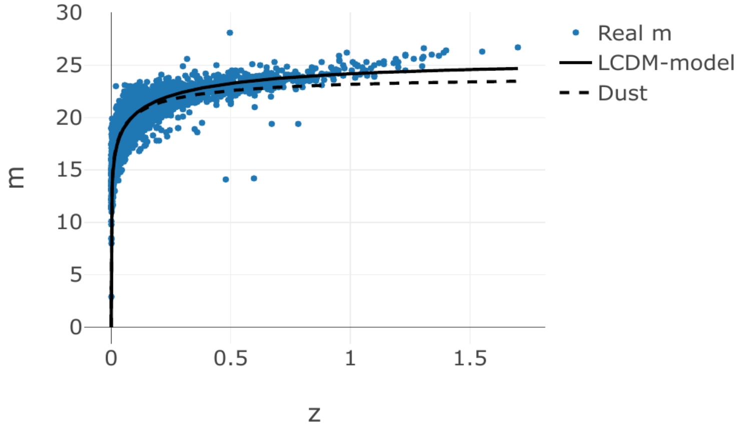

Figure 1. The dependence of the magnitude on the redshift for the LCDM-model and the Friedmann model

The dependencies of $H(z)$ are shown in figure 2, and the dependencies of the scale factor and Hubble parameter on time are shown in figures 3–5.

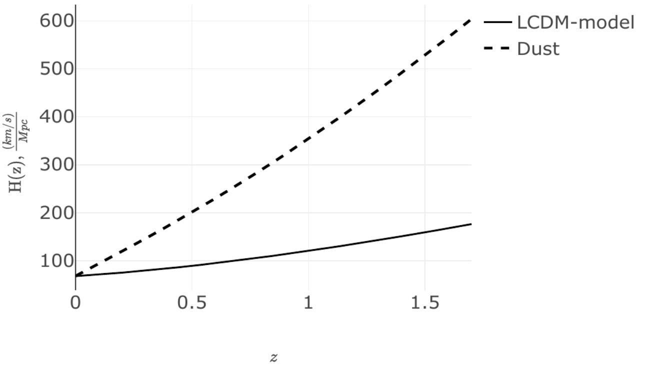

Figure 2. Dependence of the Hubble parameter on the redshift for the LCDM-model and the Friedmann model

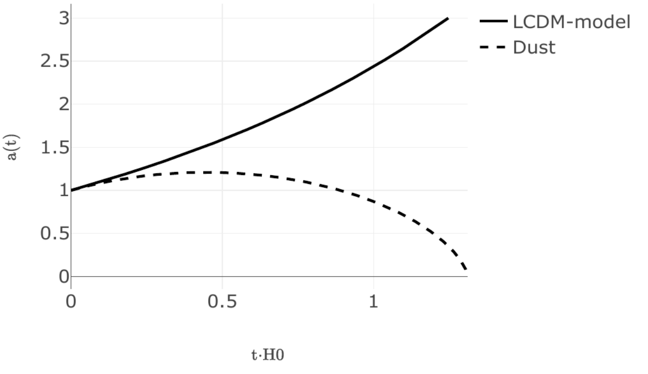

Figure 3. Dependence of the scale factor on time for the LCDM-model and the Friedmann model

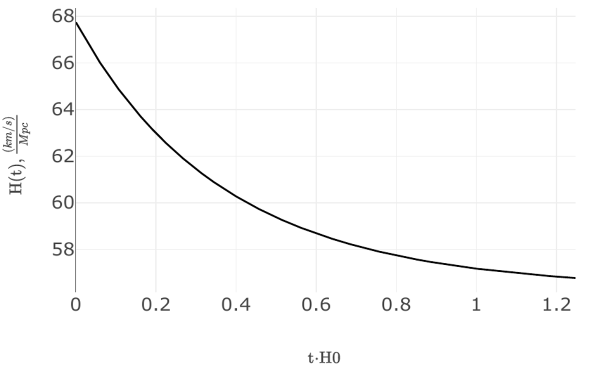

Figure 4. Dependence of the Hubble parameter on time for the LCDM-model

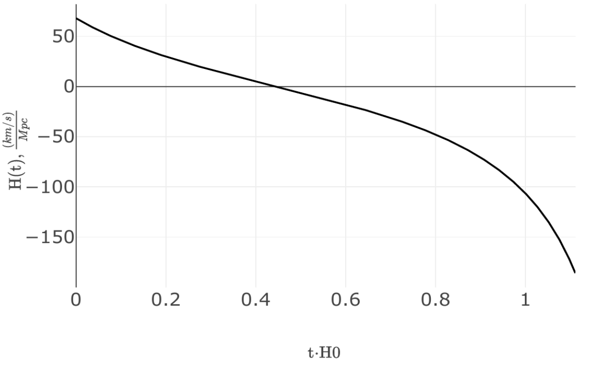

Figure 5. Dependence of the Hubble parameter on time for the Friedmann model

Table 3

Metrics for the quality of models for all z

RMSE	MSE	Model
1.4438	1.0478	Dust
1.4612	1.0579	LCDM

Comparison of quality metrics is shown in table 3.

Thus, it can be seen that the data from the SAI Supernovae Catalog is better described by the model of the dust Universe with a critical density of $\Omega_m = 5.7764$ than by the LCDM model. The Friedmann universe collapses after $\approx 1.3t_0$ of the age of the Universe ($t_0 = 1/H_0$). In turn, the LCDM model simulates an expanding Universe with acceleration. The explanation for this conclusion is the fact that the data contains a huge number (5359 out of 5614) of supernovae located in regions where the redshift $z < 0.5$. At scales where $z < 0.5$, the model of the Friedmann Universe is more efficient.

Let's look at this in more detail. Divide our data into two sets: the first data set will contain data with redshift values $z < 0.5$, and the second data set with redshift values $z > 0.5$.

We get approximations in these cases. The results are shown in figures 6–7.

Quality metrics in tables 4–5.

Table 4

Metrics for the quality of models at $z < 0.5$

RMSE	MSE	Model
1.4502	1.0506	Dust
1.4803	1.0781	LCDM

Table 5

Metrics for the quality of models at $z > 0.5$

RMSE	MSE	Model
1.0888	0.7285	Dust
0.9602	0.6001	LCDM

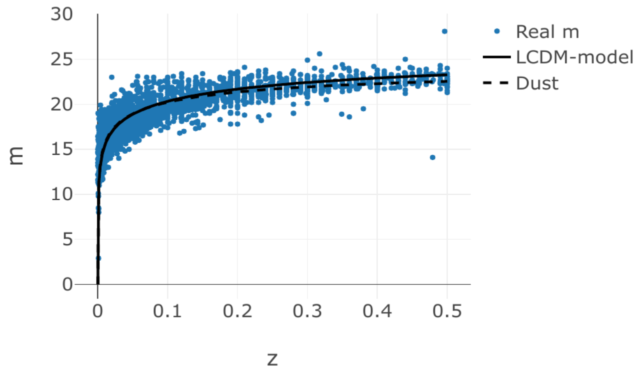

Figure 6. The dependence of the magnitude on the redshift for the LCDM-model and the Friedmann model at $z < 0.5$

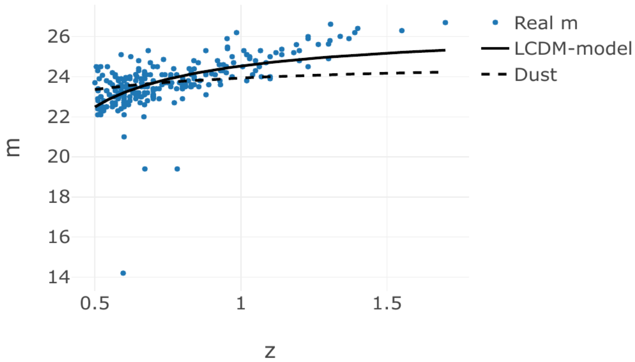

Figure 7. The dependence of the magnitude on the redshift for the LCDM-model and the Friedmann model at $z > 0.5$

5. Discussion

As we have already mentioned, thanks to modern technology astronomers and astrophysicists have been gathering a huge number of data about the past and present of our Universe. These helps us not only understand the past of the Universe, but also predict the future. Based on those data cosmologists try to construct the theoretical models and compare them with the observational data obtain the reliable one. Here we did the same within the scope of the simplest models. The idea was to construct some mathematical models which can be used in future for more complicated and realistic cases.

6. Conclusions

It was shown that the efficiency of the LCDM-model based on dark energy dominance better describes the behavior of the Universe at large scales than the model based on dust dominance. This corresponds to the modern view of the evolution of the Universe.

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Применение моделей Фридмана для описания эволюции Вселенной на основе данных SAI Supernovae Catalog

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В последние годы благодаря современным и изощрённым технологиям астрономы и астрофизики смогли заглянуть вглубь Вселенной. Полученные при этом данные ставят перед космологами новые проблемы. Одна из проблем заключается в разработке адекватной и достаточной теории. Другая проблема заключается в сопоставлении теоретических результатов с результатами наблюдений. В настоящем докладе в рамках изотропной и однородной космологической модели Фридмана–Леметра–Робертсона–Уолкера (FLRW) мы изучаем эволюцию Вселенной, заполненной пылью или космологической постоянной. Причина рассмотрения этих моделей заключается в том, что нынешняя Вселенная удивительно однородна и изотропна в больших масштабах. Мы также сравниваем наши результаты с данными из каталога SAI Supernovae Catalog. Поскольку данные наблюдений даны в терминах постоянной Хаббла (H) и красного смещения (z), мы перепишем соответствующие уравнения в виде функций от z. Задача состоит в том, чтобы найти набор параметров для математической модели изотропной и однородной Вселенной, который лучше всего соответствует астрономическим данным, полученным при изучении сверхновых: звёздная величина (m), красное смещение (z).

Ключевые слова: фитирование, космология, Вселенная Фридмана, анализ данных