Discrete & Continuous Models
& Applied Computational Science
2019, 27 (1) 5–20
http://journals.rudn.ru/miph

Queuing theory

Research article

UDC 519.872
DOI: 10.22363/2658-4670-2019-27-1-5-20

Heavy outgoing call asymptotics for retrial queue with two way communication and multiple types of outgoing calls

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(received: December 7, 2018; accepted: October 21, 2019)

In this paper, we consider a single server queueing model $M|M|1|N$ with two types of calls: incoming calls and outgoing calls, where incoming calls arrive at the server according to a Poisson process. Upon arrival, an incoming call immediately occupies the server if it is idle or joins an orbit if the server is busy. From the orbit, an incoming call retries to occupy the server and behaves the same as a fresh incoming call. The server makes an outgoing calls after an exponentially distributed idle time. It can be interpreted as that outgoing calls arrive at the server according to a Poisson process. There are N types of outgoing calls whose durations follow N distinct exponential distributions. Our contribution is to derive the asymptotics of the number of incoming calls in retrial queue under the conditions of high rates of making outgoing calls and low rates of service time of each type of outgoing calls. Based on the obtained asymptotics, we have built the approximations of the probability distribution of the number of incoming calls in the system.

Key words and phrases: retrial queueing system, incoming calls, outgoing calls, asymptotic analysis method, Gaussian approximation

1. Introduction

Retrial queueing systems are characterized by the following distinctive feature: a customer who cannot receive service remains in the system and tries to occupy the server after some random delay. The pool of unsatisfied customers is called the orbit. Retrial queues have applications in telecommunication, computer networks and in daily life [1, 2].

In retrial queues idle time of the server is the downtime and it should be reduced to increase the efficiency of the system. We consider systems where operator not only receives calls from outside but also makes outgoing calls

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in the idle time. In queueing theory a model with this feature have been considered previously [3]. However, the retrial behaviour of customers is not taken into account. In call centers operators could receive arriving calls but as soon as they have free time and are standby mode they could make outgoing calls [4–7]. Systems with this server behaviour are called retrial queues with two way communication. Retrial Queues with two way communication have been studied recently [8–11]. In these papers Markovian models with two way communication were considered. Model of retrial queue with two way communication and multiple types of outgoing calls was considered by Sakurai and Phung-Duc [12]. For this model numerical algorithm of calculating joint stationary distribution of system state was obtained. Multiserver retrial queue with two way communication was studied in [13]. Recently the two way communication retrial queues with finite source [14], with server-orbit interaction [15, 16], with finite orbit [17], with breakdowns [18] and with a constant retrial rate [19] were considered.

Asymptotic analysis methods have applications in queueing theory. Nazarov, Paul and Gudkova propose an asymptotic analysis method to research $M|M|1|1$ retrial queue with two way communication under low rate of retrials condition [20]. Nazarov, Paul and Phung-Duc extended this model to MMPP$|M|1|1$ retrial queues and derived asymptotics in heavy outgoing call conditions [21].

In this paper, we consider retrial queue with two way communication and multiple types of outgoing calls. We assume that each type of outgoing calls has different rate and service times follow distinct exponential distributions. The main aim of this paper is to derive asymptotics for the model in two limit conditions: i) high rate of outgoing calls and ii) low service rate of outgoing calls. In both cases, the number of incoming calls in the system increases.

The rest of the current paper is organized as follows. In Section 2 and 3, we describe the model in detail and preliminaries for later asymptotic analysis. In Section 4 and 5, we present our main contribution to the model. In Section 6 we show the ranges of parameters under which our approximations are usable. Section 7 is devoted to concluding remarks.

2. Model and preliminaries

2.1. Model description

Figure 1 shows the structure of the model.

We consider a single server retrial queue with two way communication and multiple types of outgoing calls. Incoming calls arrive at the system according to a Poisson process with rate λ and try to occupy the server for an exponentially distributed time with rate μ_1. Incoming calls that find the server busy join the orbit and repeat their request for service after an exponentially distributed time with rate σ. When the server is idle it makes an outgoing call of type n in an exponentially distributed time with rate α_n. There are N types of outgoing calls whose durations follow N distinct exponential distributions. We assume that the durations of outgoing calls of type n follow the exponential distribution with rate μ_n.

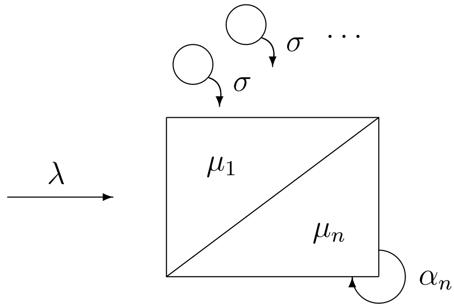

Figure 1. Markovian retrial queue with two way communication and multiple types of outgoing calls

2.2. Problem definition

Let $k(t)$ denote the state of the server at the time $t \geqslant 0$,

$$k(t) = \begin{cases} 0, & \text{if the server is idle,} \\ 1, & \text{if an incoming call is in service,} \\ n, & \text{if an outgoing call of type } n \text{ is in service, } n = \overline{2, N+1}. \end{cases}$$

Let $i(t)$ denote the number of incoming calls in the system at the time t. It is easy to see that process $\{k(t),\, i(t)\}$ forms a continuous time Markov chain. We assume that the Markov chain is ergodic and the stationary distribution of $\{k(t),\, i(t)\}$ exists.

Let $P\{k(t) = k,\, i(t) = i\} = P_k(i)$ denote the stationary probability distribution of the system state which is the unique solution of Kolmogorov system of equations:

$$\begin{cases} -\left[\lambda + i\sigma + \sum\limits_{n=2}^{N+1} \alpha_n\right] P_0(i) + \mu_1 P_1(i+1) + \sum\limits_{n=2}^{N+1} \mu_n P_n(i) = 0, \\ -(\lambda + \mu_1)P_1(i) + \lambda P_1(i-1) + \lambda P_0(i-1) + i\sigma P_0(i) = 0, \\ -(\lambda + \mu_n)P_n(i) + \lambda P_n(i-1) + \alpha_n P_0(i) = 0, \quad n = \overline{2, N+1}. \end{cases} \tag{1}$$

Let $H_k(u)$ denote the partial characteristic functions $H_k(u) = \sum\limits_{i=0}^{\infty} e^{jui} P_k(i)$, $k = \overline{0, N+1}$, where $j = \sqrt{-1}$. Multiplying equations of system (1) by e^{jui} and taking the sum over i yields

$$\begin{cases} -\left[\lambda + \sum\limits_{n=2}^{N+1} \alpha_n\right] H_0(u) + j\sigma H_0'(u) + \mu_1 e^{-ju} H_1(u) + \sum\limits_{n=2}^{N+1} \mu_n H_n(u) = 0, \\ -(\lambda + \mu_1)H_1(u) + \lambda e^{ju} H_1(u) + \lambda e^{ju} H_0(u) - j\sigma H_0'(u) = 0, \\ -(\lambda + \mu_n)H_n(u) + \lambda e^{ju} H_n(u) + \alpha_n H_0(u) = 0, \quad n = \overline{2, N+1}. \end{cases} \tag{2}$$

The characteristic function $H(u)$ of the number of incoming calls in the retrial queue is expressed through partial characteristic functions $H_k(u)$ by $H(u) = \sum\limits_{k=0}^{N+1} H_k(u)$. The main content of this paper is the solution of system (2) by using an asymptotic analysis methods in two limit conditions: of the high rate of making outgoing calls and the low rate of service time of outgoing calls.

3. Prelimit analysis

In this section, we obtain expressions for the stationary distribution using the characteristic functions. First, we derive explicit expression for the characteristic function $H(u)$ of the number of incoming calls in the system.

Theorem 1. *Explicit expression for the characteristic function $H(u)$ of the number of incoming calls in $M|M|1|N$ retrial queue is given as follows:*

$$H(u) = \frac{1}{1+\nu_1}\left(1 + \sum_{n=2}^{N+1} \frac{\alpha_n}{\mu_n + \lambda(1-e^{ju})}\right) \times \\ \times \left[\frac{1-\rho}{1-\rho e^{ju}}\right]^{\frac{\lambda}{\sigma}(1+\nu_2)+1} \prod_{n=2}^{N+1} \left[\frac{1-p_n}{1-p_n e^{ju}}\right]^{\frac{\alpha_n(\theta_n-\lambda)}{\sigma\theta_n}},$$

where

$$\rho = \frac{\lambda}{\mu_1}, \quad \nu_1 = \sum_{k=2}^{N+1} \frac{\alpha_k}{\mu_k}, \quad \nu_2 = \sum_{k=2}^{N+1} \frac{\alpha_k}{\theta_k},$$

$$p_n = \frac{\lambda}{\mu_n + \lambda}, \quad \theta_n = \lambda + \mu_n - \mu_1, \quad n = \overline{2, N+1}.$$

Proof. From equations 2 and 3 of the system (2) we obtain expressions for partial characteristic functions:

$$H_1(u) = \frac{\lambda e^{ju}}{\mu_1 + \lambda(1-e^{ju})} H_0(u) - \frac{j\sigma}{\mu_1 + \lambda(1-e^{ju})} H_0'(u), \tag{3}$$

$$H_n(u) = \frac{\alpha_n}{\mu_n + \lambda(1-e^{ju})} H_0(u), \ n = \overline{2, N+1}. \tag{4}$$

Substituting this equations into the first equation of the system (2), we find that

$$H_0'(u) = j\frac{\lambda}{\sigma}\left[\frac{\lambda e^{ju}}{\mu_1 - \lambda e^{ju}} + \frac{\mu_1 + \lambda(1-e^{ju})}{\mu_1 - \lambda e^{ju}} \sum_{n=2}^{N+1} \frac{\alpha_n e^{ju}}{\mu_n + \lambda(1-e^{ju})}\right] H_0(u). \tag{5}$$

The solution of this differential equation is given by

$$H_0(u) = r_0 \left[\frac{1-\rho}{1-\rho e^{ju}}\right]^{\frac{\lambda}{\sigma}(1+\nu_2)} \prod_{n=2}^{N+1} \left[\frac{1-p_n}{1-p_n e^{ju}}\right]^{\frac{\alpha_n(\theta_n-\lambda)}{\sigma(\theta_n)}}, \qquad (6)$$

where $\rho = \dfrac{\lambda}{\mu_1}$, $r_0 = H_0(0) = P\{k(t) = 0\}$, $\nu_2 = \sum\limits_{k=2}^{N+1} \dfrac{\alpha_k}{\theta_k}$, $p_n = \dfrac{\lambda}{\mu_n+\lambda}$, $\theta_n = \lambda + \mu_n - \mu_1$, $n = \overline{2, N+1}$.

Substituting $u = 0$ into the system (2) yields:

$$\begin{cases} -\left(\lambda + \sum\limits_{n=2}^{N+1} \alpha_n\right) r_0 + j\sigma \left. H_0'(u)\right|_{u=0} + \sum\limits_{k=1}^{N+1} \mu_k r_k = 0, \\ -\mu_1 r_1 + \lambda r_0 - j\sigma \left. H_0'(u)\right|_{u=0} = 0, \\ -\mu_n r_n + \alpha_n r_0 = 0, \quad n = \overline{2, N+1}, \end{cases} \qquad (7)$$

where expression for $\left. H_0'(u)\right|_{u=0}$ can be obtained substituting $u = 0$ into (5).

It follows from equations 2 and 3 of the system (7) that

$$r_1 = \left[\frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_1-\lambda}\left(\frac{\lambda}{\mu_1} + \sum_{n=2}^{N+1} \frac{\alpha_n}{\mu_n}\right)\right] r_0,$$

$$r_n = \frac{\alpha_n}{\mu_n} r_0, \quad n = \overline{2, N+1}.$$

Furthermore, from the normalization condition: $\sum\limits_{k=0}^{N+1} r_k = 1$, we obtain

$$r_0 = \frac{\mu_1-\lambda}{\mu_1(1+\nu_1)}, \quad r_1 = \frac{\lambda}{\mu_1}, \quad r_n = \frac{\alpha_n(\mu_1-\lambda)}{\mu_1\mu_n(1+\nu_1)}, \quad n = \overline{2, N+1},$$

where $\nu_1 = \sum\limits_{k=2}^{N+1} \dfrac{\alpha_k}{\mu_k}$. Substituting (6) into (3) and (4) and summing up results, we obtain

$$H(u) = \frac{1}{1+\nu_1}\left(1 + \sum_{n=2}^{N+1} \frac{\alpha_n}{\mu_n + \lambda(1-e^{ju})}\right) \times \\ \times \left[\frac{1-\rho}{1-\rho e^{ju}}\right]^{\frac{\lambda}{\sigma}(1+\nu_2)+1} \prod_{n=2}^{N+1} \left[\frac{1-p_n}{1-p_n e^{ju}}\right]^{\frac{\alpha_n(\theta_n-\lambda)}{\sigma\theta_n}}.$$

4. Asymptotic analysis of the model under the high rate of making outgoing calls

In this section, we will investigate system (2) by asymptotic analysis method under the high rate of making outgoing calls condition. In particular, we prove that asymptotic characteristic function of the number of incoming calls in the system corresponds to Gaussian distribution.

Denoting $\alpha_n = \alpha\gamma_n$, we obtain

$$\begin{cases} -\left[\lambda + \alpha \sum\limits_{n=2}^{N+1} \gamma_n\right] H_0(u) + j\sigma H_0'(u) + \mu_1 e^{-ju} H_1(u) + \sum\limits_{n=2}^{N+1} \mu_n H_n(u) = 0, \\ -(\lambda+\mu_1)H_1(u) + \lambda e^{ju} H_1(u) + \lambda e^{ju} H_0(u) - j\sigma H_0'(u) = 0, \\ -(\lambda+\mu_n)H_n(u) + \lambda e^{ju} H_n(u) + \alpha\gamma_n H_n(u) = 0, \quad n = \overline{2, N+1}. \end{cases} \tag{8}$$

4.1. First order asymptotic

Theorem 2. *Suppose $i(t)$ is the number of incoming calls in the system of the stationary $M|M|1|N$ retrial queue with outgoing calls, then the* (9) *holds*

$$\lim_{\alpha\to\infty} E e^{jw\frac{i(t)}{\alpha}} = e^{jw\kappa_1}, \tag{9}$$

where

$$\kappa_1 = \frac{\lambda\nu_1\mu_1}{\sigma(\mu_1-\lambda)}, \quad \nu_1 = \sum_{n=2}^{N+1} \frac{\gamma_n}{\mu_n}. \tag{10}$$

Proof. We denote $\alpha = 1/\varepsilon$ in the system (8), and introduce the following notations

$$u = \varepsilon w, \ H_0(u) = \varepsilon F_0(w,\varepsilon), \quad H_k(u) = F_k(w,\varepsilon), \quad k = \overline{1, N+1},$$

in order to get the following system

$$\begin{cases} -(\lambda\varepsilon + \sum\limits_{n=2}^{N+1} \gamma_n) F_0(w,\varepsilon) + j\sigma \dfrac{\partial F_0(w,\varepsilon)}{\partial w} + \mu_1 e^{-jw\varepsilon} F_1(w,\varepsilon) + \\ \qquad + \sum\limits_{n=2}^{N+1} \mu_n F_n(w,\varepsilon) = 0, \\ -(\lambda+\mu_1)F_1(w,\varepsilon) + \lambda e^{jw\varepsilon} F_1(w,\varepsilon) + \lambda e^{jw\varepsilon} \varepsilon F_0(w,\varepsilon) - \\ \qquad - j\sigma \dfrac{\partial F_0(w,\varepsilon)}{\partial w} = 0, \\ -(\lambda+\mu_n)F_n(w,\varepsilon) + \lambda e^{jw\varepsilon} F_n(w,\varepsilon) + \\ \qquad + \gamma_n F_0(w,\varepsilon) = 0, \quad n = \overline{2, N+1}. \end{cases} \tag{11}$$

Summing up equations of system (11), we obtain

$$\lambda\varepsilon F_0(w,\varepsilon) + (\lambda - \mu_1 e^{-jw\varepsilon})F_1(w,\varepsilon) + \lambda \sum_{n=2}^{N+1} F_n(w,\varepsilon) = 0. \tag{12}$$

Considering the limit as $\varepsilon \to 0$ in the system (11) and equation (12), then we will get

$$\begin{cases} -\sum\limits_{n=2}^{N+1} \gamma_n F_0(w) + j\sigma F_0'(w) + \sum\limits_{k=1}^{N+1} \mu_k F_k(w) = 0, \\ -\mu_1 F_1(w) - j\sigma F_0'(w) = 0, \\ -\mu_n F_n(w) + \gamma_n F_0(w) = 0, \quad n = \overline{2, N+1}, \\ -(\mu_1 - \lambda)F_1(w) + \lambda \sum\limits_{n=2}^{N+1} F_n(w) = 0. \end{cases} \tag{13}$$

We propose to get the solution of the system (13) in the form of

$$F_k(w) = \Phi(w) r_k, \quad k = \overline{0, N+1}. \tag{14}$$

Here r_k, $k = \overline{1, N+1}$ is the probability of the server state k; r_0 has no sense of probability, since the probability that the server will be in the zero state as $\alpha \to \infty$ is zero:

$$\begin{cases} -\sum\limits_{n=2}^{N+1} \gamma_n r_0 + j\sigma \dfrac{\Phi'(w)}{\Phi(w)} r_0 + \sum\limits_{k=1}^{N+1} \mu_k r_k = 0, \\ -\mu_1 r_1 - j\sigma \dfrac{\Phi'(w)}{\Phi(w)} r_0 = 0, \\ -\mu_n r_n + \gamma_n r_0 = 0, \quad n = \overline{2, N+1}, \\ -(\mu_1 - \lambda) r_1 + \lambda \sum\limits_{n=2}^{N+1} r_n = 0. \end{cases} \tag{15}$$

As the relation $j\dfrac{\Phi'(w)}{\Phi(w)}$ does not depend on w, the function is obtained in the following form $\Phi(w) = \exp\{jw\kappa_1\}$, which coincides with (9). The value of the parameter κ_1 will be defined below. We rewrite the system (15) in the form

$$\begin{cases} -\sum\limits_{n=2}^{N+1}\gamma_n r_0 - \kappa_1 r_0\sigma + \sum\limits_{k=1}^{N+1}\mu_k r_k = 0, \\ -\mu_1 r_1 + \kappa_1 r_0 \sigma = 0, \\ -\mu_n r_n + \gamma_n r_0 = 0, \quad n = \overline{2, N+1}, \\ -(\mu_1 - \lambda) r_1 + \lambda \sum\limits_{n=2}^{N+1} r_n = 0. \end{cases} \tag{16}$$

The normalization condition for stationary server state probability distribution is $\sum\limits_{k=1}^{N+1} r_k = 1$. We have

$$\begin{cases} -\mu_n r_n + \gamma_n r_0 = 0, \ n = \overline{2, N+1}, \\ -(\mu_1 - \lambda) r_1 + \lambda \sum\limits_{n=2}^{N+1} r_n = 0, \\ \sum\limits_{k=1}^{N+1} r_k = 1. \end{cases} \tag{17}$$

The solution of the system (17) is given by

$$r_0 = \frac{\mu_1 - \lambda}{\mu_1 \nu}, \quad r_1 = \frac{\lambda}{\mu_1}, \quad r_n = \frac{\gamma_n(\mu_1 - \lambda)}{\mu_n \mu_1 \nu_1}, \quad n = \overline{2, N+1}, \tag{18}$$

where $\nu_1 = \sum\limits_{n=2}^{N+1} \frac{\gamma_n}{\mu_n}$. Substituting (18) into system (16), we obtain an equation for κ_1, which coincides with (10).

The first order asymptotic i.e. Theorem 2, only defines the mean asymptotic value $\kappa_1\alpha$ of a number of incoming calls in the system in prelimit situation of $\alpha \to \infty$. For more detailed research of number $i(t)$ of incoming calls in the system let's consider the second order asymptotic. □

4.2. Second order asymptotic

Theorem 3. *In the context of Theorem 2 the following equation is true*

$$\lim_{\alpha\to\infty} E \exp\left\{ jw \frac{\frac{i(t)}{\alpha} - \kappa_1}{\sqrt{\alpha}} \right\} = e^{\frac{(jw)^2}{2}\kappa_2}, \tag{19}$$

where

$$\kappa_2 = \frac{\lambda}{\sigma} \cdot \frac{\mu_1(\mu_1 - \lambda)(\lambda\nu_2 + \nu_1) + \lambda^2 \nu_1}{(\mu_1 - \lambda)^2}, \quad \nu_1 = \sum_{n=2}^{N+1} \frac{\gamma_n}{\mu_n}, \quad \nu_2 = \sum_{n=2}^{N+1} \frac{\gamma_n}{\mu_n^2}. \tag{20}$$

Proof. We introduce the following notations in the system (8)

$$H_k(u) = \exp\{ju\alpha\kappa_1\}H_k^{(2)}(u), \quad k = \overline{0, N+1}, \tag{21}$$

and we get

$$\begin{cases} -\left(\lambda + \alpha\sum_{n=2}^{N+1}\gamma_n + \alpha\sigma\kappa_1\right)H_0^{(2)}(u) + j\sigma\dfrac{dH_0^{(2)}(u)}{du} + \\ \qquad + \mu_1 e^{-ju}H_1^{(2)}(u) + \sum_{n=2}^{N+1}\mu_n H_n^{(2)}(u) = 0, \\ -(\lambda + \mu_1)H_1^{(2)}(u) + \lambda e^{ju}H_1^{(2)}(u) + (\lambda e^{ju} + \alpha\sigma\kappa_1)H_0^{(2)}(u) - \\ \qquad - j\sigma\dfrac{dH_0^{(2)}(u)}{du} = 0, \\ -(\lambda + \mu_n)H_n^{(2)}(u) + \lambda e^{ju}H_n^{(2)}(u) + \\ \qquad + \alpha\gamma_n H_0^{(2)}(u) = 0, \quad n = \overline{2, N+1}. \end{cases} \tag{22}$$

Denoting $\alpha = 1/\varepsilon^2$, and introducing the following notations

$$\begin{gathered} u = w\varepsilon, \quad H_0^{(2)}(u) = \varepsilon^2 F_0^{(2)}(w,\varepsilon), \\ H_k^{(2)}(u) = F_k^{(2)}(w,\varepsilon), \quad k = \overline{1, N+1}, \end{gathered} \tag{23}$$

we obtain

$$\begin{cases} j\sigma\varepsilon\dfrac{\partial F_0^{(2)}(w,\varepsilon)}{\partial w} - \left(\sigma\kappa_1 + \lambda\varepsilon^2 + \sum_{n=2}^{N+1}\gamma_n\right)F_0^{(2)}(w,\varepsilon) + \\ \qquad + \mu_1 e^{-jw\varepsilon}F_1^{(2)}(w,\varepsilon) + \sum_{n=2}^{N+1}\mu_n F_n^{(2)}(w,\varepsilon) = 0, \\ -(\lambda + \mu_1)F_1^{(2)}(w,\varepsilon) + \lambda e^{jw\varepsilon}F_1^{(2)}(w,\varepsilon) + \\ \quad + (\lambda e^{jw\varepsilon}\varepsilon^2 + \sigma\kappa_1)F_0^{(2)}(w,\varepsilon) - j\sigma\varepsilon\dfrac{\partial F_0^{(2)}(w,\varepsilon)}{\partial w} = 0, \\ -(\lambda + \mu_n)F_n^{(2)}(w,\varepsilon) + \lambda e^{jw\varepsilon}F_n^{(2)}(w,\varepsilon) + \\ \qquad + \gamma_n F_0^{(2)}(w,\varepsilon) = 0, \quad n = \overline{2, N+1}. \end{cases} \tag{24}$$

Summing up equations of the system (24), we obtain

$$\lambda\varepsilon^2 F_0^{(2)}(w,\varepsilon) + (\lambda - \mu_1 e^{-jw\varepsilon})F_1^{(2)}(w,\varepsilon) + \lambda\sum_{n=2}^{N+1}F_n^{(2)}(w,\varepsilon) = 0. \tag{25}$$

Our idea is to seek for a solution of the system (24) and equation (25) in the form

$$F_k^{(2)}(w,\varepsilon) = \Phi_2(w)\{r_k + jw\varepsilon f_k\} + o(\varepsilon^2), \quad k=\overline{0,N+1}. \tag{26}$$

Substituting (26) to (24) and (25), laying out the exhibitors in tailor series and taking (16) into account, dividing these equations by ε and taking the limit as $\varepsilon \to 0$, we have

$$-\left(\sigma\kappa_1 + \sum_{n=2}^{N+1}\gamma_n\right) f_0 + \sum_{k=1}^{N+1}\mu_k f_k - \mu_1 r_1 + \sigma\frac{\Phi_2'(w)}{w\Phi(w)} r_0 = 0,$$

$$\sigma\kappa_1 f_0 - \mu_1 f_1 + \lambda r_1 \ - \sigma\frac{\Phi_2'(w)}{w\Phi(w)} r_0 = 0,$$

$$-\mu_n f_n + \lambda r_n + \gamma_n f_0 = 0, \quad n=\overline{2,N+1},$$

$$-(\mu_1-\lambda) f_1 + \lambda\sum_{n=2}^{N+1} f_n + \mu_1 r_1 = 0.$$

This equations imply that $\dfrac{\Phi_2'(w)}{w\Phi_2(w)}$ doesn't depend on w and thus the function $\Phi_2(w)$ is given in the following form

$$\Phi_2(w) = \exp\left\{\frac{(jw)^2}{2}\kappa_2\right\},$$

which coincides with (19). We have

$$\frac{\Phi_2'(w)}{w\Phi_2(w)} = -\kappa_2$$

and then we obtain the system

$$\begin{cases} -\left(\sigma\kappa_1 + \sum_{n=2}^{N+1}\gamma_n\right) f_0 + \sum_{k=1}^{N+1}\mu_k f_k = \mu_1 r_1 + \sigma\kappa_2 r_0, \\ \sigma\kappa_1 f_0 - \mu_1 f_1 = -\lambda r_1 \ - \sigma\kappa_2 r_0, \\ -\mu_n f_n + \gamma_n f_0 = -\lambda r_n, \ n=\overline{2,N+1}, \\ -(\mu_1-\lambda) f_1 + \lambda\sum_{n=2}^{N+1} f_n = -\mu_1 r_1. \end{cases} \tag{27}$$

Substituting values (18) into the system (27), we have

$$f_n = \frac{\gamma_n}{\mu_n} f_0 + \frac{\lambda(\mu_1-\lambda)\gamma_n}{\mu_1\mu_n^2\nu_1}, \quad n=\overline{2,N+1},$$

$$f_1 = \frac{\lambda\nu_1}{\mu_1-\lambda} f_0 + \frac{\lambda^2\nu_2}{\mu_1\nu_1} + \frac{\lambda}{\mu_1-\lambda},$$

where

$$\nu_1 = \sum_{k=2}^{N+1} \frac{\gamma_k}{\mu_k}, \quad \nu_2 = \sum_{k=2}^{N+1} \frac{\gamma_k}{\mu_k^2}.$$

Substituting this expressions into equation 2 of the system (27), we obtain an equation for κ_1, which coincides with (20).

Second order asymptotic i.e. Theorem 3, shows that the asymptotic probability distribution of the number $i(t)$ of incoming calls in the system is Gaussian with mean asymptotic $\kappa_1\alpha$ and variance $\kappa_2\alpha$. □

5. Asymptotic analysis of the model under the low rate of service time of outgoing calls

In this section, we will investigate system (2) by asymptotic analysis method under the low rate of service time of outgoing calls condition.

Denoting $\mu_n = \mu\gamma_n$, we obtain

$$\begin{cases} -\left[\lambda + \sum\limits_{n=2}^{N+1} \alpha_n\right] H_0(u) + j\sigma H_0'(u) + \mu_1 e^{-ju} H_1(u) + \mu \sum\limits_{n=2}^{N+1} \gamma_n H_n(u) = 0, \\ -(\lambda+\mu_1)H_1(u) + \lambda e^{ju} H_1(u) + \lambda e^{ju} H_0(u) - j\sigma H_0'(u) = 0, \\ -(\lambda+\mu\gamma_n)H_n(u) + \lambda e^{ju} H_n(u) + \alpha_n H_n(u) = 0, \ n = \overline{2, N+1}. \end{cases} \tag{28}$$

Theorem 4. *Suppose $i(t)$ is a number of incoming calls in a system of stationary $M|M|1|N$ retrial queue with two way communication, then the following equation is true*

$$H(u) = \lim_{\mu\to 0} E e^{jw\mu i(t)} = \frac{1}{\nu_1} \sum_{n=2}^{N+1} \frac{\alpha_n}{\gamma_n - jw\lambda} \prod_{n=2}^{N+1} \left(1 - jw\frac{\lambda}{\gamma_n}\right)^{-\frac{\mu_1\alpha_n}{\sigma(\mu_1-\lambda)}}, \tag{29}$$

where $\nu_1 = \sum\limits_{n=2}^{N+1} \frac{\alpha_n}{\gamma_n}$.

Proof. We denote $\mu = \varepsilon$, let's substitute the following in the system (28)

$$u = w\varepsilon, \ H_0(u) = \varepsilon F_0(w,\varepsilon), \quad H_k(u) = F_k(w,\varepsilon), \quad k = \overline{1, N+1}.$$

We will get the system

$$\begin{cases} -\left(\lambda+\sum\limits_{n=2}^{N+1}\alpha_n\right)\varepsilon F_0(w,\varepsilon)+j\sigma\dfrac{\partial F_0(w,\varepsilon)}{\partial w}+ \\ \qquad\qquad +\mu_1 e^{-jw\varepsilon}F_1(w,\varepsilon)+\varepsilon\sum\limits_{n=2}^{N+1}\gamma_n F_n(w,\varepsilon)=0, \\ -(\lambda+\mu_1)F_1(w,\varepsilon)-j\sigma\dfrac{\partial F_0(w,\varepsilon)}{\partial w}+\lambda e^{jw\varepsilon}F_1(w,\varepsilon)+ \\ \qquad\qquad\qquad\qquad +\lambda\varepsilon e^{jw\varepsilon}F_0(w,\varepsilon)=0, \\ -(\lambda+\varepsilon\gamma_n)F_n(w,\varepsilon)+\lambda e^{jw\varepsilon}F_n(w,\varepsilon)+ \\ \qquad\qquad +\alpha_n\varepsilon F_0(w,\varepsilon)=0, \quad n=\overline{2,N+1}. \end{cases} \tag{30}$$

Considering the limit as $\varepsilon \to 0$ in the system (30) then we will get

$$-j\sigma F_0'(w)-\mu_1F_1(w)=0,\ j\sigma F_0'(w)+\mu_1F_1(w)=0. \tag{31}$$

Summing up equations of the system (30) we have

$$\lambda\varepsilon F_0(w,\varepsilon)+(\lambda-\mu_1e^{-jw\varepsilon})F_1(w,\varepsilon)+\lambda\sum_{n=2}^{N+1}\gamma_nF_n(w,\varepsilon)=0. \tag{32}$$

Laying out the exhibitors in tailor series, dividing equations by ε and taking the limit as $\varepsilon \to 0$, taking (31) into account, we obtain

$$-\left(\lambda+\sum_{n=2}^{N+1}\alpha_n\right)F_0(w)-jw\mu_1F_1(w)+\sum_{n=2}^{N+1}\gamma_nF_n(w)=0,$$

$$-j\sigma F_0'(w)-\mu_1F_1(w)=0,$$

$$(\lambda jw-\gamma_n)F_n(w)+\alpha_nF_0(w)=0, \quad n=\overline{2,N+1}$$

$$-(\mu_1-\lambda)F_1(w)+\lambda\sum_{n=2}^{N+1}F_n(w)=0.$$

From the last system of equations we have

$$F_n(w)=\frac{\alpha_n}{\gamma_n-jw\lambda}F_0(w), \tag{33}$$

$$F_1(w)=\frac{\lambda}{\mu_1-\lambda}\sum_{n=2}^{N+1}F_n(w). \tag{34}$$

Then

$$F_1(w) = \frac{\lambda}{\mu_1 - \lambda} F_0(w) \sum_{n=2}^{N+1} \frac{\alpha_n}{\gamma_n - jw\lambda}. \tag{35}$$

Substituting (35) into (31), we obtain

$$F_0'(w) = j\frac{\lambda\mu_1}{\sigma(\mu_1 - \lambda)} F_0(w) \sum_{n=2}^{N+1} \frac{\alpha_n}{\gamma_n - jw\lambda}.$$

The solution of differential equation is given by

$$F_0(w) = C \prod_{n=2}^{N+1} \left(1 - jw\frac{\lambda}{\gamma_n}\right)^{-\frac{\mu_1\alpha_n}{\sigma(\mu_1-\lambda)}}, \tag{36}$$

where C is an integration constant and its value will be obtained later. We denote asymptotic characteristic function $\sum\limits_{k=1}^{N+1} F_k(w) = \Phi(w)$. Substituting (36) into (33) and (34), we obtain

$$F_1(w) = \frac{\lambda}{\mu_1 - \lambda} \sum_{k=2}^{N+1} \frac{\alpha_k}{\gamma_k - jw\lambda} C \prod_{k=2}^{N+1} \left(1 - jw\frac{\lambda}{\gamma_k}\right)^{-\frac{\mu_1\alpha_k}{\sigma(\mu_1-\lambda)}},$$

$$F_n(w) = \frac{\alpha_n}{\gamma_n - jw\lambda} C \prod_{k=2}^{N+1} \left(1 - jw\frac{\lambda}{\gamma_k}\right)^{-\frac{\mu_1\alpha_k}{\sigma(\mu_1-\lambda)}}, \quad n = \overline{2, N+1}.$$

Summing up equations, we have

$$\Phi(w) = \frac{\mu_1}{\mu_1 - \lambda} C \prod_{n=2}^{N+1} \left(1 - jw\frac{\lambda}{\gamma_n}\right)^{-\frac{\mu_1\alpha_n}{\sigma(\mu_1-\lambda)}} \sum_{n=2}^{N+1} \frac{\alpha_n}{\gamma_n - jw\lambda}.$$

Using condition $\Phi(0) = 1$, we obtain

$$C = \frac{\mu_1 - \lambda}{\mu_1\nu_1}, \quad \text{where} \quad \nu_1 = \sum_{n=2}^{N+1} \frac{\alpha_n}{\gamma_n}.$$

We obtain the characteristic function (29). □

6. Approximation accuracy

The accuracy of the approximation $P^{(2)}(i)$ is defined by using Kolmogorov range $\Delta_2 = \max\limits_{0\leqslant i\leqslant N} \left|\sum\limits_{\nu=0}^{i} (P(\nu) - P^{(2)}(\nu))\right|$, which represents the difference between distributions $P(i)$ and $P^{(2)}(i)$, where $P(i)$ is obtained by using inverse

Fourier transform for the characteristic function of the $M|M|1|N$ retrial queue and the approximation $P^{(2)}(i)$ is given by obtained asymptotics. We consider $N = 3$, $\lambda = 0.2$, $\mu_1 = 1$ and $\sigma = 1$ for Tables 1 and 2.

Table 1

Kolmogorov range, $\mu_2 = 2,\ \mu_3 = 3,\ \mu_4 = 4,\ \gamma_2 = 1,\ \gamma_3 = 2,\ \gamma_4 = 3$

	$\alpha = 3$	$\alpha = 5$	$\alpha = 10$	$\alpha = 50$	$\alpha = 100$
Δ_2	0.066	0.043	0.023	0.01	0.007

Table 2

Kolmogorov range, $\gamma_2 = 2,\ \gamma_3 = 3,\ \gamma_4 = \ 4,\ \alpha_2 = 1,\ \alpha_3 = 2,\ \alpha_4 = 3$

	$\mu = 0.05$	$\mu = 0.035$	$\mu = 0.02$	$\mu = 0.01$
Δ_2	0.059	0.044	0.026	0.014

7. Conclusions

In this paper, we have considered retrial queue with two way communication with multiple types of outgoing calls. We have found characteristic function of the number of incoming calls in the system. We have found the first and the second order asymptotics of the number of calls in the system under the condition of the high rate of making outgoing calls. Based on the obtained asymptotics we have built the Gaussian approximation of the probability distribution of the number of incoming calls in the system. We have found asymptotic characteristic function of the number of incoming calls in retrial queue under the condition of the low service rate of outgoing calls. In future we plan to consider this retrial queueing system under other asymptotic conditions.

Acknowledgments

This research was financially supported by Russian Foundation for Basic Research (the Agreement number 18-01-00277).

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For citation:

A. Nazarov, S. Paul, O. Lizyura, Heavy outgoing call asymptotics for retrial queue with two way communication and multiple types of outgoing calls, Discrete and Continuous Models and Applied Computational Science 27 (1) (2019) 5–20. DOI: 10.22363/2658-4670-2019-27-1-5-20.

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