Физика

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Entangled Solitons and Einstein–Podolsky–Rosen Correlations

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Stochastic realization of the wave function in quantum mechanics, with the inclusion of soliton representation of extended particles, is discussed. Entangled solitons construction being introduced in the nonlinear spinor field model, the Einstein–Podolsky–Rosen (EPR) spin correlation is calculated and shown to coincide with the quantum mechanical one for the 1/2–spin particles.

Key words and phrases: entangled solitons, stochastic representation, spinor field model.

1. Introduction. Wave–particle dualism and solitons

As a first motivation for introducing stochastic representation of the wave function let us consider the de Broglie plane wave

$$\psi = A\mathrm{e}^{-\mathrm{i}kx} = A\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}(\mathbf{kr})} \tag{1}$$

for a free particle with the energy ω, momentum $\mathbf{k}$, and mass m, when the relativistic relation

$$k^2 = \omega^2 - \mathbf{k}^2 = m^2 \tag{2}$$

holds (in natural units $\hbar = c = 1$).

Suppose, following L. de Broglie [1] and A. Einstein [2], that the structure of the particle is described by a regular bounded function $u(t,\mathbf{r})$, which is supposed to satisfy some nonlinear equation with the Klein–Gordon linear part. Let $\ell_0 = 1/m$ be the characteristic size of the soliton solution $u(t,\mathbf{r})$ moving with the velocity $\mathbf{v} = \mathbf{k}/\omega$.

Now it is worth-while to underline the remarkable fact behind this research [3], namely, the possibility to represent the de Broglie wave (1) as the sum of solitons located at nodes of a cubic lattice with the spacing $a \gg \ell_0$:

$$A\mathrm{e}^{-\mathrm{i}kx} = \sum_{\mathbf{d}} u(t,\mathbf{r}+\mathbf{d}), \tag{3}$$

where $\mathbf{d}$ marks the positions of lattice nodes. To show the validity of (3) one can take into account the asymptotic behavior of the soliton in its tail region:

$$u(x) = \int \mathrm{d}^4k\,\mathrm{e}^{-\mathrm{i}kx} g(k)\delta(k^2 - m^2) \tag{4}$$

and then use the well-known formula

$$\sum_{\mathbf{d}} \mathrm{e}^{\mathrm{i}(\mathbf{k}\,\mathbf{d})} = \left(\frac{2\pi}{a}\right)^3 \delta(\mathbf{k}), \tag{5}$$

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implying that

$$A = \left(\frac{2\pi}{a}\right)^3 \frac{g(m)}{2m}.$$

The formula (3) gives a simple illustration of the wave–particle dualism, showing that the de Broglie wave characterizes the assemblage of particles–solitons.

2. D. Bohm's principle of nonlinear resonance and its gravitational mechanism

As a point of departure we consider the following problem posed by D. Bohm. Many years ago he discussed in his book [4] the possible relation between the wave–particle dualism in quantum mechanics and nonlinearity of fundamental equations in future theory of elementary particles. To represent the line of D. Bohm's thought, let us consider in Minkowsky space–time a simple scalar field model given by the Lagrangian density

$$\mathcal{L} = \partial_i\phi^* \partial_j\phi\, \eta^{ij} - (mc/\hbar)^2\, \phi^*\phi + F\left(\phi^*\phi\right). \tag{6}$$

Here ϕ designates complex scalar field, $i,\ j = 0, 1, 2, 3$; $\eta^{ij} = \mathrm{diag}(1, -1, -1, -1)$, and the nonlinear function $F(s)$ behaves at $s \to 0$ as s^n, $n > 1$, to guarantee the existence of particle-like solutions to the corresponding field equations, that is describing localized regular configurations possessing finite energy. In particular, the choice $F(s) = g\, s^{3/2}$, $g > 0$, in (6) corresponds to the well-known Synge model [5], which is popular in nuclear physics and admits stationary radial solutions of the form

$$\phi_0 = u(r) \exp(-\mathrm{i}\omega t), \quad r = |\mathbf{r}|\,. \tag{7}$$

The radial function $u(r)$ in (7) is regular and exponentially decreases at space infinity, thus implying the finiteness of the energy

$$E = \int \mathrm{d}^3x\, T_0^0(\phi_0), \tag{8}$$

where T_j^i stands for the energy–momentum tensor of the field model in question. Moreover, it can be shown that the unnodal configuration, for which $u(r)$ ¿0, turns out to be stable in the Liapunov's sense, if the charge of the configuration is fixed [6]. This fact implies the existence of slightly perturbed soliton solutions similar to (7):

$$\phi = \phi_0 + \xi(t, \mathbf{r}). \tag{9}$$

It should be stressed that the perturbation ξ in (9) appears to be small with respect to ϕ_0 in the region of soliton's localization only, though in the "tail" region of the soliton (i.e. far from its center) the function ϕ_0 is small, so one can put $\phi = \xi$.

D. Bohm posed the following question: Does there exist any nonlinear field model, for which the asymptotic behavior of the perturbed soliton solution, at large distances from the soliton's center, would represent the oscillations with the characteristic frequency $\omega = E/\hbar$? In other words, for the model in question the principal Fourier amplitude of the field $\phi \approx \xi$ at large distances $r \to \infty$ should correspond to the frequency ω related to the soliton's energy (8) via the Planck–de Broglie formula

$$E = \hbar\omega. \tag{10}$$

This property will be called **the Bohm's principle of nonlinear resonance**.

As one can see from (6), the field equation at space infinity, where $\phi \to 0$, reduces to the linear Klein–Gordon equation

$$\left(\Box - (mc/\hbar)^2\right)\phi = 0. \tag{11}$$

Therefore, the relation (10) can be satisfied for the solitons with the single energy $E = mc^2$, determined by the fixed mass m represented in (6). Thus, we conclude that the universality of the Planck–de Broglie relation (10) appears to be broken for the model (6), that forces us to modify the latter one. Taking into account that the frequency in (10) is determined by the mass of the localized system, it seems natural to use in the new modified model the proper gravitational field of the soliton–particle, in view of the fact that its asymptotic behavior at space infinity is also determined by the mass of the system. Finally, it is suggested to search for the answer to the Bohm's question in the self-consistent gravitational theory [7, 8].

The new model will be described by the Lagrangian density $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_m$, where $\mathcal{L}_g = c^4 R/(16\pi G)$ corresponds to the Einstein gravitational theory and $\mathcal{L}_m$ is written as follows:

$$\mathcal{L}_m = \partial_i \phi^* \partial_j \phi\, g^{ij} - I(g_{ij})\phi^*\phi + F(\phi^*\phi). \tag{12}$$

The crucial point in this scheme is the constructing of the invariant $I(g_{ij})$, which should depend on the metric tensor g_{ij} of the Riemannian space–time in such a manner that in the vicinity of the soliton with a mass m the following relation took place:

$$\lim_{r\to\infty} I(g_{ij}) = (mc/\hbar)^2. \tag{13}$$

It can be easily seen that due to (13) one finds at space infinity the universal equation (11), which is valid for the soliton configuration with an arbitrary mass m.

To show the existence of the invariant I with the property (13), one could construct it through the Riemann curvature tensor R_{ijkl} and its covariant derivatives $R_{ijkl;n}$:

$$I = (I_1^4/I_2^3)c^6\hbar^{-2}G^{-2}, \tag{14}$$

where G stands for the Newton gravitational constant and invariants I_1, I_2 have the form:

$$I_1 = R_{ijkl}R^{ijkl}/48, \quad I_2 = -R_{ijkl;n}R^{ijkl;n}/432.$$

Calculating R_{ijkl} and invariants I_1, I_2 via the Schwarzschild metric at large distance r from the soliton's center, that seems reasonable for the island-like systems, one finds

$$I_1 = G^2m^2/(c^4r^6)\,; \quad I_2 = G^2m^2/(c^4r^8). \tag{15}$$

Thus, the relations (14) and (15) imply the desirable property (13) and the validity of the Bohm's principle of nonlinear resonance in its gravitational realization, that is the Planck–de Broglie wave–particle dualism relation (10) holds for all massive particles described by regular localized field configurations.

Now the next problem arises: to prove the consistency of the Einstein–de Broglie solitonian scheme, complemented by the Bohm's nonlinear resonance principle, with the main axioms of quantum mechanics. This problem was discussed in the works [9, 10] and it was shown that in the limit of point-like particles the main quantum postulates could be retained. In particular, it turned out that on the base of solitonian field configurations one could build the analog of the probability amplitude (wave function) and the mean values of physical observables could be calculated as scalar products in a suitable Hilbert space with the stochastic properties.

3. Random Hilbert space

The shortest way to get the stochastic representation of quantum mechanics is modify the formula (3). This can be easily performed if one admits that the locations of solitons' centers are not regular nodes of the cubic lattice but some randomly chosen points. To realize this prescription, suppose that a field ϕ describes n particles–solitons and has the form

$$\phi(t,\mathbf{r}) = \sum_{k=1}^{n} \phi^{(k)}(t,\mathbf{r}), \tag{16}$$

where

$$\operatorname{supp}\phi^{(k)} \cap \operatorname{supp}\phi^{(k')} = 0, \quad k \neq k',$$

and the same for the conjugate momenta

$$\pi(t,\mathbf{r}) = \partial\mathcal{L}/\partial\phi_t = \sum_{k=1}^{n} \pi^{(k)}(t,\mathbf{r}), \quad \phi_t = \partial\phi/\partial t.$$

Let us define the auxiliary functions

$$\varphi^{(k)}(t,\mathbf{r}) = \frac{1}{\sqrt{2}}(\nu_k\phi^{(k)} + \mathrm{i}\pi^{(k)}/\nu_k) \tag{17}$$

with the constants ν_k satisfying the normalization condition

$$\hbar = \int \mathrm{d}^3x\, |\varphi^{(k)}|^2. \tag{18}$$

Now we define the analog of the wave function in the configurational space $\mathbb{R}^{3n} \ni \mathbf{x} = \{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$ as

$$\Psi_N(t,\mathbf{r}_1,\ldots,\mathbf{r}_n) = (\hbar^n N)^{-1/2} \sum_{j=1}^{N} \prod_{k=1}^{n} \varphi_j^{(k)}(t,\mathbf{r}_k), \tag{19}$$

where $N \gg 1$ stands for the number of trials (observations) and $\varphi_j^{(k)}$ is the one-particle function (7) for the j th trial.

Now we intend to show that the quantity

$$\rho_N = \frac{1}{(\Delta\vee)^n} \int\limits_{(\Delta\vee)^n\subset\mathbb{R}^{3n}} \mathrm{d}^{3n}x\, |\Psi_N|^2\,,$$

where $\Delta\vee$ is the elementary volume which is supposed to be much greater than the proper volume of the particle ${\ell_0}^3 = \vee_0 \ll \Delta\vee$, plays the role of coordinate probability density. To this end let us calculate the following integral:

$$(\Delta\vee)^n\, \rho_N \equiv \int\limits_{(\Delta\vee)^n} \mathrm{d}^{3n}x\, |\Psi_N|^2 = (\hbar^n N)^{-1} \left(\sum_{i=1}^{N} a_{ii} + \sum_{i\neq j=1}^{N} a_{ij} \right),$$

where the denotation is used

$$a_{ij} = \tfrac{1}{2} \prod_{k=1}^{n} \int_{\Delta\vee} \mathrm{d}^3x\, \left(\varphi_i^{*(k)} \varphi_j^{(k)} + \varphi_j^{*(k)} \varphi_i^{(k)} \right).$$

Taking into account (19), one gets

$$(\Delta\vee)^n\, \rho_N = (\hbar^n N)^{-1} \left(\hbar^n \Delta N + S\right), \quad S = \sum_{i\neq j} a_{ij}, \tag{20}$$

with ΔN standing for the number of trials for which the centers of particles–solitons were located in $(\Delta\vee)^n$.

It is worth-while to remark that due to independence of trials and arbitrariness of initial data and, in particular, of the phases of the functions $\varphi_i^{(k)}$, one can consider the entities a_{ij} for $i \neq j$ as independent random variables with zero mean values. This fact permits to use the Chebyshev's inequality [11] for estimating the probability of the events, for which $|S|$ surpasses $\hbar^n \Delta N$:

$$P\left(|S| > \hbar^n \Delta N\right) \leqslant \left(\hbar^n \Delta N\right)^{-2} \left\langle S^2 \right\rangle. \tag{21}$$

On the other hand, in view of trials' independence one gets

$$\left\langle S^2 \right\rangle = \sum_{i \neq j} \left\langle a_{ij}^2 \right\rangle. \tag{22}$$

Now one can take into account that the wave packets $\varphi_i^{(k)}$ are effectively overlapped if their centers belong to the proper volume domain $\vee_0$. This property permits to deduce from (19) and (22) the estimate

$$\left\langle S^2 \right\rangle \leqslant \alpha^n \hbar^{2n} \frac{\Delta N}{(\Delta \vee)^n} {\vee_0}^n \Delta N, \tag{23}$$

where $\alpha \sim 1$ is the "packing" factor for the nearest neighbors. Inserting (23) into (22), one finds the following estimate:

$$P\left(|S| > \hbar^n \Delta N\right) < \left(\alpha \vee_0 / \Delta \vee\right)^n \ll 1. \tag{24}$$

Applying the estimate (24) to (20), one can state that with the probability close to unity the following relation holds:

$$\left(\Delta \vee\right)^n \rho_N = \Delta N / N, \tag{25}$$

signifying that the construction (19) plays the role of the probability amplitude for the coordinate distribution of solitons' centers, with ρ_N in (25) being the corresponding probability density.

Now let us consider the measuring procedure for some observable A corresponding, due to E. Noether's theorem, to the symmetry group generator $\hat{M}_A$. For example, the momentum $\mathbf{P}$ is related with the generator of space translation $\hat{M}_P = -\,i\,\nabla$, the angular momentum $\mathbf{L}$ is related with the generator of space rotation $\hat{M}_L = \mathbf{J}$ and so on. As a result one can represent the classical observable A_j for the j–th trial in the form

$$A_j = \int \mathrm{d}^3 x\, \pi_j \mathrm{i} \hat{M}_A \phi_j = \sum_{k=1}^{n} \int \mathrm{d}^3 x\, \varphi_j^{*(k)} \hat{M}_A^{(k)} \varphi_j^{(k)}.$$

The corresponding mean value is

$$\mathbb{E}(A) \equiv \frac{1}{N} \sum_{j=1}^{N} A_j = \frac{1}{N} \sum_{j=1}^{N} \sum_{k=1}^{n} \int \mathrm{d}^3 x\, \varphi_j^{*(k)} \hat{M}_A^{(k)} \varphi_j^{(k)} = \\ = \int \mathrm{d}^{3n} x\, \Psi_N^* \hat{A} \Psi_N + O\left(\frac{\vee_0}{\Delta \vee}\right), \tag{26}$$

where the Hermitian operator $\hat{A}$ reads

$$\hat{A} = \sum_{k=1}^{n} \hbar \hat{M}_A^{(k)}. \tag{27}$$

Thus, up to the terms of the order $\vee_0 / \Delta \vee \ll 1$, we obtain the standard quantum mechanical rule (26) for the calculation of mean values [9, 12].

It is interesting to underline that the solitonian scheme in question contains also the well-known spin–statistics correlation [8]. Namely, if $\varphi_j^{(k)}$ is transformed under the rotation by irreducible representation $D^{(J)}$ of $SO(3)$, with the weight J, then the transposition of two identical extended particles is equivalent to the relative 2π–rotation of $\varphi_j^{(k)}$, that gives the multiplication factor $(-1)^{2J}$ in Ψ_N. To show this property, suppose that our particles are identical, i.e. their profiles $\varphi_j^{(k)}$ may differ in phases only. Therefore, the transposition of the particles with the centers at $\mathbf{r}_1$ and $\mathbf{r}_2$ means the π–rotation of 2–particle configuration around the median axis of the central vector line $\mathbf{r}_1 - \mathbf{r}_2$. However, due to extended character of the particles, to restore the initial configuration, one should perform additional proper π–rotations of the particles. The latter operation being equivalent to the relative 2π–rotation of particles, one concludes that it results in aforementioned multiplication of Ψ_N by $(-1)^{2J}$. Under the natural supposition that the weight J is related with the spin of particles–solitons, one infers that the many–particles wave function (19) should be symmetrical under the transposition of the two identical particles if the spin is integer, but antisymmetrical if the spin is half-integer (the Pauli principle).

Thus, we conclude that in the solitonian scheme the spin–statistics correlation stems from the extended character of particles–solitons. However, the particles in quantum mechanics being considered as point-like ones, it appears inevitable to include the transpositional symmetry of the wave function as the first principle (cf. Hartree–Fock receipt for Fermions).

It can be also proved that Ψ_N up to the terms of order $\vee_0/\Delta\vee$ satisfies the standard Schrödinger equation [8]. To this end it is worth-while to underline that, in accordance with the Bohm's nonlinear resonance principle (13), in the vicinity of the k–th particle the Klein–Gordon equation (11) with the particle's mass m_k is satisfied. However, at large distances the same equation (11) is valid but with the mass M, equal to the total mass of the system. In view of this fact, it is useful to divide the field configuration $\varphi^{(k)}$ into two parts as follows:

$$\varphi^{(k)} = \varphi_0^{(k)} + \varphi_\infty^{(k)}, \tag{28}$$

where $\varphi_0^{(k)}$ describes the nearest structure (highly decreasing function) and $\varphi_\infty^{(k)}$ describes the far one (slightly decreasing function). According to (11), in the proper reference frames of the k–th particle and of the total system respectively, one finds the following time behavior of these functions:

$$\varphi_0^{(k)} \sim \mathrm{e}^{-\mathrm{i} m_k c^2 t/\hbar}, \quad \varphi_\infty^{(k)} \sim \mathrm{e}^{-\mathrm{i} M c^2 t/\hbar}. \tag{29}$$

Inserting (28) in (19), one gets for $r_j \to \infty$

$$\prod_{k=1}^{n} \varphi^{(k)} = \prod_{k=1}^{n} \left(\varphi_0^{(k)} + \varphi_\infty^{(k)}\right) \approx \varphi_\infty^{(k)} \prod_{k\neq j} \varphi_0^{(k)}. \tag{30}$$

In view of (29) and (30) one concludes that at $r_j \to \infty$

$$\Psi_N \sim \mathrm{e}^{-\mathrm{i} M c^2 t/\hbar}. \tag{31}$$

On the other hand, given the field Hamiltonian $H[\phi, \pi]$ of the system, one can write the field equations in the canonical form, that results in the evolution law of $\varphi^{(k)}$:

$$\mathrm{i}\partial_t \varphi^{(k)} = \delta H / \delta \varphi^{*(k)}. \tag{32}$$

Therefore, combining (19) and (32), one gets the evolution equation for Ψ_N:

$$\mathrm{i}\hbar\,\partial_t \Psi_N = \hbar \sum_{k=1}^{n} \sum_{j=1}^{N} \frac{\delta H}{\delta \varphi_j^{*(k)}} \frac{\partial \Psi_N}{\partial \varphi_j^{(k)}} \equiv \hat{H} \Psi_N, \tag{33}$$

which has the standard quantum mechanical form with some generalized Hamilton operator $\hat{H}$. As follows from (31), the operator $\hat{H}$ has the sense of the total energy operator of the system in question. Taking into account the estimate (24), one can ascertain that with the probability close to the unity the equation (33) is equivalent to some linear evolution equation for the probability amplitude [9].

Now we prove that in the nonrelativistic limit this equation should coincide with the Schrödinger equation for the system of n particles. In fact, according to (11) in the vicinity of the k–th particle the following equation holds:

$$\Box\varphi^{(k)} = (m_k c/\hbar)^2 \varphi^{(k)} + U_k(\phi, \pi),$$

which after the substitution

$$\varphi^{(k)} = u^{(k)} \mathrm{e}^{-\mathrm{i} m_k c^2 t/\hbar}$$

reduces, in the nonrelativistic limit, to the equation

$$\mathrm{i}\hbar\, \partial_t u^{(k)} \approx -\frac{\hbar^2}{2m_k} \varDelta_k u^{(k)} + U'_k,$$

where U'_k stands for an effective interaction potential. Therefore, the function

$$\psi_N = \varPsi_N \exp\left(\sum_{k=1}^{n} \mathrm{i} m_k c^2 t/\hbar\right)$$

satisfies the standard n–particle Schrödinger equation.

Now it is worth-while to discuss the evidence of wave properties of particles in solitonian scheme. To verify the fact that solitons can really possess wave properties, the *gedanken* diffraction experiment with individual electrons–solitons was realized. Solitons with some velocity were dropped into a rectilinear slit, cut in the impermeable screen, and the transverse momentum was calculated which they gained while passing the slit, with the width of the latter significantly exceeded the size of the soliton. As a result, the picture of distribution of the centers of scattered solitons was restored on the registration screen, by considering their initial distribution to be uniform over the transverse coordinate. It was clarified that though the center of each soliton fell into a definite place of the registration screen (depending on the initial soliton profile and the point of crossing the plane of the slit by the soliton's center), the statistical picture in many ways was similar to the well-known diffraction distribution in optics, i.e. the Fresnel's picture at short distances from the slit and the Fraunhofer's one at large distances [13, 14].

Various aspects of the fulfillment of the quantum mechanics correspondence principle for the Einstein–de Broglie's solitonian model were discussed in the works [8, 9, 12]. In these papers it was shown that in the framework of the solitonian model all quantum postulates were regained in the limit of point particles, so that from the physical fields one can build the amplitude of probability and the average can be calculated as a scalar product in the Hilbert space by introducing the corresponding quantum operators for observables. The fundamental role of the gravitational field in the de Broglie–Einstein solitonian scheme was discussed in [8, 15]. The solitonian model of the hydrogen atom was developed in [10, 16]. The dynamics of solitons in external fields was discussed in the paper [17].

As a result we obtain the stochastic realization (19) of the wave function $\varPsi_N$ which can be considered as an element of the random Hilbert space $\mathcal{H}_{\mathrm{rand}}$ with the scalar product

$$(\psi_1, \psi_2) = \mathbb{M}(\psi_1^* \psi_2), \tag{34}$$

with $\mathbb{M}$ standing for the expectation value. As a rude simplification one can admit that the averaging in (34) is taken over random characteristics of particles–solitons, such as their positions, velocities, phases, and so on. It is important to underline once more that the correspondence with the standard quantum mechanics is retained only in the point–particle limit ($\Delta\vee \gg \vee_0$) for $N\to\infty$. To show this [9,12], one can apply the central limit theorem stating that for $N\to\infty$ the wave function $\Psi_N(t,\mathbf{x})$ behaves as the Gaussian random field with the variance

$$\sigma^2 = \rho(t,\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^{3n}, \tag{35}$$

where $\rho(t,\mathbf{x})$ stands for the probability density (partition function) of solitons' centers in $\mathbb{R}^{3n}$.

Random Hilbert spaces being widely exploited in mathematical statistics [18], for quantum applications they were first used by N. Wiener in [19]. To illustrate the line of Wiener's argument, we recall the general scheme of introducing various representations in quantum mechanics.

Let $|\psi\rangle$ be a state vector in the Hilbert space $\mathcal{H}$ and $\hat{A}$ be a self-conjugate operator with the spectrum $\sigma(\hat{A})$. Then the a–representation is given by the wave function

$$\psi(a) = \langle a|\psi\rangle,$$

where

$$\hat{A}|a\rangle = a|a\rangle, \quad a\in\sigma(\hat{A}).$$

In particular, the famous Schrödinger coordinate q–representation is given by the wave function

$$\psi(q) = \langle q|\psi\rangle = \sum_n \langle q|n\rangle\langle n|\psi\rangle, \tag{36}$$

with $|n\rangle$ being some complete set of state vectors in $\mathcal{H}$.

Wiener considered the real Brownian process $x(s,\alpha)$ in the interval $[0,1]\ni s$, where $\alpha\in[0,1]$ is the generalized number of the Brownian trajectory and the correlation reads

$$\int_0^1 \mathrm{d}\alpha\, x(s,\alpha)x(s',\alpha) = \min(s,s'). \tag{37}$$

To obtain the quantum mechanical description, Wiener defined the complex Brownian process

$$z(s|\alpha,\beta) = \frac{1}{\sqrt{2}}\left[x(s,\alpha) + \mathrm{i}\, y(s,\beta)\right]; \quad \alpha,\beta\in[0,1], \tag{38}$$

and using the natural mapping $\mathbb{R}^3\to[0,1]$, for the particle in $\mathbb{R}^3$, constructed the stochastic representation of the wave function along similar lines as in (36):

$$\langle\alpha,\beta|\psi\rangle = \int\limits_{s\in[0,1]} \mathrm{d}z(s|\alpha,\beta)\psi(s), \tag{39}$$

with the obvious unitarity property

$$\int_0^1 \mathrm{d}s\,|\psi(s)|^2 = \iint\limits_{[0,1]^2} \mathrm{d}\alpha\,\mathrm{d}\beta|\langle\alpha,\beta|\psi\rangle|^2$$

stemming from (37).

4. Entangled solitons and EPR correlations

In the sequel we shall consider the special case of two–particles configurations ($n = 2$), corresponding to the singlet state of two 1/2–spin particles. In quantum mechanics these states are described by the spin wave function of the form

$$\psi_{12} = \frac{1}{\sqrt{2}} \left(|1 \uparrow\rangle \otimes |2 \downarrow\rangle - |1 \downarrow\rangle \otimes |2 \uparrow\rangle \right) \tag{40}$$

and are known as **entangled states**. The arrows in (40) signify the projections of spin $\pm 1/2$ along some fixed direction. In the case of the electrons in the famous Stern–Gerlach experiment this direction is determined by that of an external magnetic field. If one chooses two different Stern–Gerlach devices, with the directions **a** and **b** of the magnetic fields, denoted by the unit vectors **a** and **b** respectively, one can measure the correlation of spins of the two electrons by projecting the spin of the first electron on **a** and the second one on **b**. Quantum mechanics gives for the spin correlation function the well-known expression

$$P(\mathbf{a},\, \mathbf{b}) = \psi^{+}_{12} (\sigma \mathbf{a}) \otimes (\sigma \mathbf{b}) \psi_{12}, \tag{41}$$

where σ stands for the vector of Pauli matrices σ_i, $i = 1, 2, 3$. Putting (40) into (41), one easily gets

$$P(\mathbf{a},\, \mathbf{b}) = -(\mathbf{ab}). \tag{42}$$

The formula (42) characterizes the spin correlation in the Einstein–Podolsky–Rosen entangled singlet states and is known as the EPR–correlation. As was shown by J. Bell [20], the correlation (42) can be used as an efficient criterium for distinguishing the models with the local (point-like) hidden variables from those with the nonlocal ones. Namely, for the local-hidden-variables theories the EPR–correlation (42) is broken.

It would be interesting to check the solitonian model, shortly described in the beforehand points, by applying to it the EPR–correlation criterium. To this end let us first describe the 1/2–spin particles as solitons in the nonlinear spinor model of Heisenberg–Ivanenko type considered in the works [21, 22]. The soliton in question is described by the relativistic 4–spinor field φ of stationary type

$$\varphi = \begin{bmatrix} u \\ v \end{bmatrix} \mathrm{e}^{-\mathrm{i}\omega t}, \tag{43}$$

satisfying the equation

$$\left(\mathrm{i}\gamma^k \partial_k - \ell_0^{-1} + \lambda(\bar{\varphi}\varphi) \right) \varphi = 0, \tag{44}$$

where u and v denote 2–spinors, k runs Minkowsky space indices 0, 1, 2, 3; ℓ_0 stands for some characteristic length (the size of the particle–soliton), λ is self-coupling constant, $\bar{\varphi} \equiv \varphi^{+}\gamma^0$, γ^k are the Dirac matrices. The stationary solution to the equation (44) can be obtained by separating variables in spherical coordinates r, ϑ, α via the substitution

$$u = \frac{1}{\sqrt{4\pi}} f(r) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v = \frac{\mathrm{i}}{\sqrt{4\pi}} g(r) \sigma_r \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{45}$$

where $\sigma_r = (\sigma \mathbf{r})/r$. Inserting (45) into (44), one finds

$$\begin{aligned} &\frac{\omega}{c} u + \mathrm{i}(\sigma\nabla)v - \ell_0^{-1} u + \frac{\lambda}{4\pi} \left(f^2 - g^2 \right) u = 0, \\ &\frac{\omega}{c} v + \mathrm{i}(\sigma\nabla)u - \ell_0^{-1} v + \frac{\lambda}{4\pi} \left(f^2 - g^2 \right) v = 0. \end{aligned}$$

In view of (45) one gets

$$\mathrm{i}(\sigma\nabla)v = -\frac{1}{\sqrt{4\pi}}\left(g' + \frac{2}{r}g\right)\begin{bmatrix}1\\0\end{bmatrix},$$
$$\mathrm{i}(\sigma\nabla)u = -\frac{\mathrm{i}}{\sqrt{4\pi}}f'\sigma_r\begin{bmatrix}1\\0\end{bmatrix}.$$

Finally, one derives the following ordinary differential equations for the radial functions $f(r)$ and $g(r)$:

$$\left(g' + \frac{2}{r}g\right) = \left(\frac{\omega}{c} - \ell_0^{-1}\right)f + \frac{\lambda}{4\pi}\left(f^2 - g^2\right)f,$$
$$-f' = \left(\frac{\omega}{c} + \ell_0^{-1}\right)g + \frac{\lambda}{4\pi}\left(f^2 - g^2\right)g.$$

As was shown in the papers [21, 22], these equations admit regular solutions, if the frequency parameter ω belongs to the interval

$$0 < \omega < c/\ell_0. \tag{46}$$

The behavior of the functions $f(r)$ and $g(r)$ at $r \to 0$ is as follows:

$$g(r) = C_1 r, \quad f = C_2, \quad f' \to 0,$$

where C_1, C_2 denote some integration constants. The behavior of solutions far from the center of the soliton, i.e. at $r \to \infty$, is given by the relations:

$$f = \frac{A}{r}\mathrm{e}^{-\nu r}, \quad g = -\frac{f'}{B},$$

where

$$\nu = \left(\ell_0^{-2} - \omega^2/c^2\right)^{1/2}, \quad B = \ell_0^{-1} + \omega/c.$$

If one chooses the free parameters ℓ_0 and λ of the model to satisfy the normalization condition (similar to (19))

$$\int \mathrm{d}^3x\, \varphi^+\varphi = \int\limits_0^\infty \mathrm{d}r\, r^2\left(f^2 + g^2\right) = \hbar, \tag{47}$$

then the spin of the soliton reads

$$\mathbf{S} = \int \mathrm{d}^3x\, \varphi^+\mathbf{J}\varphi = \frac{\hbar}{2}\mathbf{e}_z, \tag{48}$$

where $\mathbf{e}_z$ denotes the unit vector along the Z–direction, $\mathbf{J}$ stands for the angular momentum operator

$$\mathbf{J} = -\mathrm{i}[\mathbf{r}\nabla] + \frac{1}{2}\sigma\otimes\sigma_0, \tag{49}$$

and σ_0 is the unit 2×2–matrix.

Now it is worth-while to show the positiveness of the energy E of the 1/2–spin soliton. The energy E is given by the expression

$$E = c\int \mathrm{d}^3x\left[-\mathrm{i}\varphi^+(\alpha\nabla)\varphi + \ell_0^{-1}\bar{\varphi}\varphi - \frac{\lambda}{2}(\bar{\varphi}\varphi)^2\right], \tag{50}$$

where $\alpha = \sigma \otimes \sigma_1$. The positiveness of the functional (50) emerges from the virial identities characteristic for the model in question. In fact, the equation for the stationary solution (43) can be derived from the variational principle based on the Lagrangian of the system

$$L = -E + \int \mathrm{d}^3 x\, \omega \varphi^+ \varphi. \tag{51}$$

Performing the two-parameters scale transformation of the form $\varphi(x) \to \alpha\varphi(\beta x)$, one can derive from (51) and the variational principle $\delta L = 0$ the following two virial identities, which are valid for any regular stationary solution to the field equation (44):

$$\int \mathrm{d}^3 x \left[\mathrm{i}\frac{2}{3}\varphi^+(\alpha\nabla)\varphi + \frac{\omega}{c}\varphi^+\varphi - \ell_0^{-1}\bar{\varphi}\varphi + \frac{\lambda}{2}(\bar{\varphi}\varphi)^2\right] = 0, \tag{52}$$

$$\int \mathrm{d}^3 x \left[\mathrm{i}\varphi^+(\alpha\nabla)\varphi + \frac{\omega}{c}\varphi^+\varphi - \ell_0^{-1}\bar{\varphi}\varphi + \lambda(\bar{\varphi}\varphi)^2\right] = 0. \tag{53}$$

Using (52) and (53), one can express some sign-changing integrals through those of definite sign:

$$\int \mathrm{d}^3 x \left[-\mathrm{i}\frac{1}{3}\varphi^+(\alpha\nabla)\varphi\right] = \frac{\lambda}{2}\int \mathrm{d}^3 x\, (\bar{\varphi}\varphi)^2, \tag{54}$$

$$\int \mathrm{d}^3 x \left[\ell_0^{-1}\bar{\varphi}\varphi + \frac{\lambda}{2}(\bar{\varphi}\varphi)^2\right] = \frac{\omega}{c}\int \mathrm{d}^3 x\, \varphi^+\varphi. \tag{55}$$

Using the identities (54) and (55), one can represent the energy (50) of the soliton as follows:

$$E = c\int \mathrm{d}^3 x \left[\ell_0^{-1}\bar{\varphi}\varphi + \lambda(\bar{\varphi}\varphi)^2\right] = \omega\int \mathrm{d}^3 x\, \varphi^+\varphi = \hbar\omega, \tag{56}$$

where the normalization condition (47) was taken into account. Thus, one concludes, in the connection with (46) and (56), that the energy of the stationary spinor soliton (43) in the nonlinear model (44) turns out to be positive. Moreover, one can see that (56) is equivalent to the Planck–de Broglie wave–particle dualism relation (11).

Now let us construct the two–particles singlet configuration on the base of the soliton solution (43). First of all, in analogy with (40), one constructs the entangled solitons configuration endowed with the zero spin:

$$\varphi_{12} = \frac{1}{\sqrt{2}}\left[\varphi_1^{\uparrow} \otimes \varphi_2^{\downarrow} - \varphi_1^{\downarrow} \otimes \varphi_2^{\uparrow}\right], \tag{57}$$

where $\varphi_1^{\uparrow}$ corresponds to (45) with $\mathbf{r} = \mathbf{r}_1$, and $\varphi_2^{\downarrow}$ emerges from the above solution by the substitution

$$\mathbf{r}_1 \to \mathbf{r}_2, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

that corresponds to the opposite projection of spin on the Z–axis. In virtue of the orthogonality relation for the states with the opposite spin projections, one easily derives the following normalization condition for the entangled solitons configuration (57):

$$\int \mathrm{d}^3 x_1 \int \mathrm{d}^3 x_2\, \varphi_{12}^+\varphi_{12} = \hbar^2. \tag{58}$$

Now it is not difficult to find the expression for the stochastic wave function (20) for the singlet two–solitons state:

$$\Psi_N\left(t, \mathbf{r}_1, \mathbf{r}_2\right) = \left(\hbar^2 N\right)^{-1/2}\sum_{j=1}^{N}\varphi_{12}^{(j)}, \tag{59}$$

where $\varphi_{12}^{(j)}$ corresponds to the entangled soliton configuration in the j th trial.

Our final step is the calculation of the spin correlation (41) for the singlet two–soliton state. In the light of the fact that the operator σ in (41) corresponds to the twice angular momentum operator (49), one should calculate the following expression:

$$P'(\mathbf{a},\mathbf{b}) = \mathbb{M} \int \mathrm{d}^3x_1 \int \mathrm{d}^3x_2\, \Psi_N^+ 2\,(\mathbf{J}_1\mathbf{a}) \otimes 2\,(\mathbf{J}_2\mathbf{b})\,\Psi_N, \qquad (60)$$

where $\mathbb{M}$ stands for the averaging over the random phases of the solitons. Inserting (59) and (49) into (60), using the independence of trials $j \neq j'$ and taking into account the relations:

$$J_+\varphi^{\uparrow} = 0, \quad J_3\varphi^{\uparrow} = \frac{1}{2}\varphi^{\uparrow}, \qquad J_-\varphi^{\uparrow} = \varphi^{\downarrow},$$
$$J_-\varphi^{\downarrow} = 0, \quad J_3\varphi^{\downarrow} = -\frac{1}{2}\varphi^{\downarrow}, \quad J_+\varphi^{\downarrow} = \varphi^{\uparrow},$$

where $J_{\pm} = J_1 \pm iJ_2$, one easily finds that

$$P'(\mathbf{a},\mathbf{b}) = -\hbar^{-2}\,(\mathbf{ab}) \left(\int\limits_0^\infty \mathrm{d}r\, r^2\left(f^2+g^2\right) \right)^2 = -\,(\mathbf{ab})\,. \qquad (61)$$

Comparing the correlations (61) and (42), one remarks their coincidence, that is the solitonian model satisfies the EPR–correlation criterium.

Conclusion

The main purpose of this paper was to find new arguments in favour of the thought that the soliton concept advocated by Einstein and de Broglie can give a consistent description of extended quantum particles. In particular, as a motivation for such a conclusion, within a framework of nonlinear spinor field model the solitonian image of 1/2–spin particles was used for constructing two–solitons singlet configuration, which permitted to calculate the spin EPR correlation. Fascinating result of this calculation was the coincidence of the quantum spin correlation with that in the solitonian scheme. This latter fact supports the hope that the solitonian scheme has many attractive features relevant to consistent theory of extended elementary particles. The search for such a theory was considered by V.V. Kuryshkin as a first task in his mighty scientific activity.

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Запутанные солитоны и корреляции Эйнштейна–Подольского–Розена

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Обсуждается стохастическая реализация волновой функции в квантовой механике на основе солитонного представления протяжённых частиц. Для построения запутанных состояний в обобщённой квантовой механике протяжённых частиц используются двухсолитонные конфигурации. Конструкция запутанных солитонов в модели нелинейного спинорного поля применяется для вычисления спиновой корреляции Эйнштейна–Подольского–Розена (ЭПР) и показывается, что она совпадает с квантовой ЭПР–корреляцией для частиц спина 1/2.