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The Analysis of Queuing System with General Service Distribution and Renovation

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We investigate the queuing system in which the losses of incoming orders due to the introduction of a special renovation mechanism are possible. The introduced queuing system consists of server with a general distribution of service time and a buffer of unlimited capacity. The incoming flow of tasks is a Poisson one. The renovation mechanism is that at the end of its service the task on the server may with some probability empty the buffer and leave the system, or with an additional probability may just leave the system. In order to study the characteristics of the system the Markov chain embedded upon the end of service times is introduced. Under the assumption of the existence of a stationary regime for the embedded Markov chain the formula for the probability generation function is obtained. With the help of the probability generation function such system characteristics as the probability of the system being empty, the average number of customers in the system, the probability of a task not to be dropped, the distribution of the service waiting time for non-dropped tasks, the average service waiting time for non-dropped requests are derived.

Key words and phrases: queuing system, renovation, general service distribution, probability characteristics

1. Introduction

Due to the study of mathematical models close to reality, there is a growing need to find new solutions and methods of queuing systems construction and analysis. One of the classical and already well-studied systems close to reality is the system $M/G/1/\infty$ with the Poisson incoming flow and general service distribution.

The described non-Markov random process — a process where the future state can not depend only on the state, viewed in the given time. The system $M/G/1/\infty$ can be investigated by a variety of different approaches and one of them is the construction of the embedded Markov chain [1].

Thanks to the Pollaczek–Khinchin formula [2] for a given stationary probability distribution some other characteristics can be also derived. These characteristics of the system $M/G/1/\infty$ with the standard service discipline FCFS (First Come, First Served) can be transferred to some other common discipline [1, 2].

However, trying to describe a real system, it is necessary to take into account the possibility of losing data in the system, for example, due to failure of an unreliable server [3] or due to arrival of some “viral” applications [4], and the other tasks in the buffer will be dropped. This situation may be investigated with the help of queuing system

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$M/G/1/\infty$ with renovation without repeated service [5–9]. Systems with renovation mechanism can be used for traffic control mechanism modeling [10].

We analyze the queueing system in which losses of the accepted customers are possible due to the so called renovation.

2. System Description

Consider $M/G/1/\infty$ queueing system with the general service time distribution $B(x)$, Poisson arrival rate λ and renovation. The renovation mechanism, due to [5,7,8], operates as follows. At the end of each service completion the customer leaving server with the (known) probability q empties the buffer and leaves the system. With the probability $p = 1 - q$ it leaves the system without having any effect on the buffer contents. If $p = 1$ one obtains the well-known $M/G/1/\infty$ queue.

As usual, if one considers the total number of customers $\{\nu_i, i \geq 0\}$ in the system just after i -th service completion, then $\{\nu_i, i \geq 0\}$ is the embedded Markov chain of the queue-length process $\{\nu(t), t \geq 0\}$. Denote the state set of the embedded Markov by $\mathcal{X} = \{0, 1, \dots\}$.

The matrix of transition probabilities for the embedded Markov chain has the form:

$$\begin{pmatrix} \beta_0 + \sum_{i=1}^{\infty} \beta_i q & \beta_1 p & \beta_2 p & \beta_3 p & \beta_4 p & \beta_5 p & \dots \\ \beta_0 + \sum_{i=1}^{\infty} \beta_i q & \beta_1 p & \beta_2 p & \beta_3 p & \beta_4 p & \beta_5 p & \dots \\ \sum_{i=0}^{\infty} \beta_i q & \beta_0 p & \beta_1 p & \beta_2 p & \beta_3 p & \beta_4 p & \dots \\ \sum_{i=0}^{\infty} \beta_i q & 0 & \beta_0 p & \beta_1 p & \beta_2 p & \beta_3 p & \dots \\ \sum_{i=0}^{\infty} \beta_i q & 0 & 0 & \beta_0 p & \beta_1 p & \beta_2 p & \dots \\ \sum_{i=0}^{\infty} \beta_i q & 0 & 0 & 0 & \beta_0 p & \beta_1 p & \dots \\ \sum_{i=0}^{\infty} \beta_i q & 0 & 0 & 0 & 0 & \beta_0 p & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

3. Stationary Distribution of the Embedded Markov Chain

Denote by $p_i, i \geq 0$, the probability, that there are i customers in the system upon service completion. Then, assuming that the stationary distribution exists, one has the following system for p_i :

$$p_0 = \left(\beta_0 + \sum_{i=1}^{\infty} \beta_i q \right) p_0 + \left(\beta_0 + \sum_{i=1}^{\infty} \beta_i q \right) p_1 + \sum_{k=2}^{\infty} \left(\sum_{i=0}^{\infty} \beta_i q \right) p_k, \quad (1)$$

$$p_i = \beta_i p p_0 + \sum_{k=1}^{i+1} p \beta_{i+1-k} p_k, \quad i \geq 1. \quad (2)$$

Here $\beta_i = \int_0^\infty \frac{(\lambda x)^i}{i!} e^{-\lambda x} dB(x)$ denotes the probability that during service time exactly i ($i \geq 0$) other customers have entered the system. It is straightforward to show, that the probability generation function (PGF)

$$P(z) = \sum_{i=0}^{\infty} p_i z^i$$

and can be written in the following form:

$$P(z) = \frac{(1-z) p p_0 \beta(\lambda - \lambda z) - z q}{p \beta(\lambda - \lambda z) - z}, \quad (3)$$

where

$$\beta(\lambda - \lambda z) = \sum_{i=0}^{\infty} \beta_i z^i = \sum_{i=0}^{\infty} z^i \int_0^\infty \frac{(\lambda x)^i}{i!} e^{-\lambda x} dB(x).$$

If $p = 1$ then (3) coincides with the Pollaczek-Khinchin formula for classic $M/G/1/\infty$ queue.

4. Performance Characteristics

Using the analytical property of $P(z)$ one can obtain the expression for the probability of system being empty upon service completion. Consider the equation

$$p \beta(\lambda - \lambda z) - z = 0.$$

It has the unique solution $0 < z_0 < 1$ for $z \in [0, 1]$. As the denominator of (3) vanishes at point $z = z_0$ then the numerator of (3) must also vanish at this point. Thus

$$(1 - z_0) p p_0 \beta(\lambda - \lambda z_0) - z_0 q = 0,$$

wherefrom it follows that

$$p_0 = \frac{q z_0}{(1 - z_0) p \beta(\lambda - \lambda z_0)}. \quad (4)$$

If $p = 1$ ($q = 1 - p = 0$) one gets the well-known expression $p_0 = 1 - \lambda b$, where b is the mean service time.

The average number N of customers in the system upon service completion is equal to

$$N = P'(1) = \frac{p}{q} (p_0 + \lambda b - 1). \quad (5)$$

Here λb is the average number of customers arrived during a single service time.

Let us denote by $p^{(\text{serv})}$ the probability that all the customers in the system just after the end of the service will not be dropped and will eventually receive service. Then

$$p^{(\text{serv})} = \frac{1}{1 - p_0} \sum_{i=1}^{\infty} p_i p^{i-1} = \frac{1}{p(1 - p_0)} (P(p) - p_0). \quad (6)$$

The $P(p)$ is the value of probability generation function (3) $P(z)$ with $z = p$.

Let us denote as $W^{(\text{serv})}(x)$ the probability, that the waiting time for the last customer in the buffer (just after the end of the service) will be less than x :

$$W^{(\text{serv})}(x) = \frac{1}{(1-p_0)p^{(\text{serv})}} \sum_{i=1}^{\infty} W_i^{(\text{serv})}(x)p_i,$$

here $W_i^{(\text{serv})}(x)$ is the probability, that the waiting time for the i -th customer in the buffer (just after the end of the service) will be less than x with the requirement that there were exactly i customers just after the end of service.

The Laplace–Stieltjes transformation of $W^{(\text{serv})}(x)$ has the form:

$$\omega^{(\text{serv})}(s) = \frac{1}{(1-p_0)p^{(\text{serv})}} \left(p_1 + \sum_{i=2}^{\infty} p_i p^{i-1} \beta(s)^{i-1} \right) = \frac{1}{(1-p_0)p^{(\text{serv})}} \frac{P(p\beta(s)) - p_0}{p\beta(s)}. \quad (7)$$

The mean waiting time of the customer which received service is equal to:

$$w^{(\text{serv})} = - \left(\omega^{(\text{serv})}(s) \right)'_{s=0} = \frac{b}{1-p_0} \left(\frac{P'(p)}{p^{(\text{serv})}} - 1 \right), \quad (8)$$

where

$$P'(p) = \frac{b}{p(\beta(\lambda q) - 1)^2} (p^2 p_0 + qp - p_0 p + \lambda p_0 p(1-p)pb - \lambda p^2 qb) = \frac{qp(1-p_0)}{(1-\beta(\lambda q))^2}.$$

5. Conclusion

The paper considers the queueing system with full renovation. Analytical expressions for the main performance characteristics are obtained. The study of $M/G/1/\infty$ queues with the general renovation as well as with renovation and re-service (due to [11]) is an open issue.

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Анализ системы массового обслуживания с рекуррентным обслуживанием и полным обновлением

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В работе исследуется система массового обслуживания, в которой возможны потери поступающих заявок из-за введённого специального механизма обновления. Система состоит из одного обслуживающего прибора с рекуррентным распределением времени обслуживания и накопителя неограниченной ёмкости, в рассматриваемую систему поступает пуассоновский поток заявок. Механизм обновления заключается в том, что в момент окончания обслуживания на приборе заявка либо может опустошить весь накопитель и покинуть систему, либо с дополнительной вероятностью просто покинуть систему. Для исследования характеристик рассматриваемой системы строится вложенная по моментам окончания обслуживания цепь Маркова. В предположении о существовании стационарного режима для построенной вложенной цепи Маркова выводится производящая функция числа заявок в системе, вероятность простоя системы, среднее число заявок в системе, вероятность отсутствия потерь, распределение времени ожидания начала обслуживания несброшенных заявок, среднее время ожидания обслуживания для несброшенной заявки.

Ключевые слова: полное обновление, система массового обслуживания, рекуррентное обслуживание, сброс заявок, вероятностные характеристики

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