Математика

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On Almost Global Half-Geodesic Parameterization

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The problem of existence of Global Half-Geodesic Surface Parameterization is considered. The problem is well known and it is yet unsolved in general case. It is known that for the twice-differentiable surfaces it has local solution. At the same time example of paraboloid of revolution proves that it is not possible in the general case to use local nets in order to construct the global half- geodesic ones. In order to solve the problem the authors follow the way leading to the construction of isothermal parameterization for the surfaces with positive first quadratic form. To this end they deduce partial differential equation for the mappings giving necessary parameterization.

In the contrast with the case of isothermal parameterization when the equation is Beltrami equation corresponding to the homogeneous elliptic system this equation is essentially non-linear one. Besides the new system admits degeneration at the points where the Jacobian of the solution is equal to zero or infinity. The speed of degeneration strongly affecting properties of the solutions is also unknown.

In order to surpass these difficulties the authors change the challenge. Instead of the geodesics covering the whole surface they propose to find the geodesics covering the surface up to the set of Hausdorff null measure. Using the theory of K-quasiconformal mappings they construct non-regular generalized solutions of non-linear Beltrami equation that nevertheless detect the necessary family of the geodesics. The constructed theory permits to study non-classical equilibrium forms of liquid drops.

Key words and phrases: first quadratic form of the parametric surface, quasiconformal mappings, generalized solution, non-linear Beltrami equation, Sobolev spaces, imbedding theorems, weak convergence of the functions, almost global half-geodesic parameterization

1. Formulation of the problem

Let $X = X(\xi, \eta)$, $(\xi, \eta) \in D$ be a continuously differentiable parametric surface and ω — its first quadratic form

$$\omega = A\mathrm{d}\xi^2 + 2B\mathrm{d}\xi\mathrm{d}\eta + C\mathrm{d}\eta^2. \tag{1}$$

Let us suppose that the following conditions are satisfied

$$A \geqslant A_0 > 0, \quad \Delta = AC - B^2 \geqslant \Delta_0 > 0. \tag{2}$$

It is well-known [1] that there exists a homeomorphism $z = z(\zeta)$, $\zeta = \xi + i\eta$ of the disk D onto itself reducing the quadratic form ω in the isothermal coordinates (x, y) to the canonical one

$$\omega = \Lambda(x, y)(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{3}$$

The function $z = z(\zeta)$ satisfies the following Beltrami equation

$$z_{\overline{\zeta}}(\zeta) = \frac{A - \sqrt{\Delta} + iB}{A + \sqrt{\Delta} - iB} z_\zeta(\zeta). \tag{4}$$

The isothermal coordinates proved to be very useful in various applications, especially in the theory of minimal surfaces.

The other canonical representation of the type

$$\omega^*(u,v) = \mathrm{d}u^2 + H\mathrm{d}v^2 \tag{5}$$

is also widely known [2].

The parameterization of the surface reducing the first quadratic form to the form (5) has the name of half-geodesic parameterization.

Locally we can introduce for the surfaces under consideration the coordinates (u, v) transforming their first quadratic form into the form (5).

Sometimes it is possible (for example for the surfaces of the negative curvature or for the analytic ones) to get this reduction over the whole surface. For the general case the problem still stands to be open.

Meanwhile sometimes the global half-geodesic parameterization is also necessary, especially in the case when the functional giving the Gauss curvature under variation is required (see for example [3–5]).

The existence of such parameterization is connected with existence of the family of the geodesics covering the whole surface.

In this article we are going to prove the existence of the family of geodesics without self-intersections covering the surfaces X we consider almost everywhere. It is sufficient, for example, for the generalization of the variational problems studied in the articles [3–5] to the case of the surfaces lacking the axial symmetry.

In order to fulfill this task we will proceed as follows. As in the case of isothermal parameterization we deduce Beltrami differential equation for the transformations resolving the problem of the existence of half-geodesic parameterization. After this using the method of successive approximations we prove the existence of the solutions of the equation. Finally, we prove that almost all level lines of the imaginary parts of the solution the equation for the inverse transformation are geodesics covering almost all the surface X.

The main difficulty we encounter on this way lies in non-linear nature of Beltrami equation we get in our case. Besides, it degenerates at the points where the Jacobian of the transformation turns to be zero or infinite.

The instrument of changing the variables turns not to be perfect but still it permits us to detect almost all the geodesics of the surface and prove that they cover it without self-intersection up to the set of null Hausdorff measure.

2. Basic equations

It is easy to prove the following theorem.

Theorem 1. *Let $X = X(x, y)$ be continuously differentiable parametric surface with the first quadratic form ω of the type (3). Let us suppose that there exists a topological sense preserving transformation $w : D \to D$, $w = u + iv$ reducing the form ω to the form ω^* from formula (5). In this case, the function w satisfies the conjugate nonlinear Beltrami equation*

$$w_{\overline{z}}(z) = \frac{\Lambda(z) - J_w(z)}{J_w(z) + \Lambda(z)}\overline{w}_z. \tag{6}$$

The inverse function $z = z(w)$, $z = x + iy$, satisfies the nonlinear Beltrami equation

$$z_{\overline{w}}(w) = -\frac{\Lambda\left(z(w)\right) J_z(w) - 1}{\Lambda\left(z(w)\right) J_z(w) + 1} z_w(w). \tag{7}$$

Besides the following equality takes place

$$H(u,v) = \Lambda\left(z(u,v)\right)\left(x_v^2 + y_v^2\right) = \Lambda^2\left(z(u,v)\right) J_z^2(w).$$

This theorem constitutes the first part of our program.

3. Construction of the solutions of the basic equations

Let us start now with the study of the second part of our program. We will prove that the solution of the conjugate non-linear Beltrami equation (6) exists. It is clear that this equation degenerates at the points where the Jacobian of the solution is equal to zero or infinity. This entails many difficulties as we cannot apply directly the well-developed theory of quasiconformal mappings with bounded characteristics.

In what follows we restrict our considerations to the twice-differentiable surfaces symmetrical relatively to a plane P. Besides, we suppose that the intersection of the surface X with the plane P is also geodesics. We denote by the letter Ξ the class of the surfaces of this type.

In order to find the needed solution we will use the method of iterations. We divide this process into two stages.

3.1. First stage

At the first stage the process of the iterations is as following. Firstly, for each number $m \in N$ fixed we construct the sequence of K-quasiconformal mappings satisfying the following equations

$$w_{\overline{z}}^{(n+1)}(m,z) = \frac{\Lambda(z) - J_{w^{(n)}}(z)}{J_{w^{(n)}}(z) + \Lambda(z)} \frac{1-\frac{1}{m}}{1+\frac{1}{m}} \overline{w}_{z}^{(n+1)}(m,z). \tag{8}$$

As the first element of the sequence we introduce into equation (8) as J_{w^0} the Jacobian of the conformal mapping w^0 of the unit disk D onto itself normalized by the correspondence of the three pairs of the boundary points

$$w^0(-1) = -1, \quad w^0(i) = p, \quad w^0(1) = 1, \quad 0 < \arg p < \pi.$$

Let

$$w^{(n)}(m,z) = u^{(n)}(m,z) + iv^{(n)}(m,z).$$

Then the functions $u^{(n+1)}(m,z)$, $v^{(n+1)}(m,z)$ satisfy the following system of the equations equivalent to the equation (8),

$$\left(\frac{1}{m} + \Lambda^{-1} J_{w^{(n)}}(m,z)\right) u_x^{(n+1)}(m,z) = \left(1 + \frac{1}{m} J_{w^{(n)}}(m,z)\right) v_y^{(n+1)}(m,z),$$

$$\left(\frac{1}{m} + \Lambda^{-1} J_{w^{(n)}}(m,z)\right) u_y^{(n+1)}(m,z) = -\left(1 + \frac{1}{m}\Lambda^{-1} J_{w^{(n)}}(m,z)\right) v_x^{(n+1)}(m,z). \tag{9}$$

Let now $z^{(n+1} = x^{(n+1)} + iy^{(n+1)}$ be the mapping inverse to the mapping $w^{(n+1)}$. We can easily prove that it satisfies the following non-linear Beltrami equation

$$z_{\overline{w}}^{(n+1)}(m,w) = -\frac{J_{w^{(n)}}\left(m, z^{(n+1)}(w)\right) - \Lambda\left(z^{(n+1)}(w)\right)}{J_{w^{(n)}}\left(m, z^{(n+1)}(w)\right) + \Lambda\left(z^{(n+1)}(w)\right)} \frac{1-\frac{1}{m}}{1+\frac{1}{m}} z_w^{(n+1)}(m,w).$$

The functions $x^{(n+1)}$, $y^{(n+1)}$ satisfy the equations

$$\frac{\frac{1}{m} J_{w^{(n)}}\left(m, z^{(n+1)}\right) + \Lambda\left(z^{(n+1)}(w)\right)}{J_{w^{(n)}}\left(m, z^{(n+1)}\right) + \frac{1}{m}\Lambda\left(z^{(n+1)}(w)\right)} x_u^{(n+1)}(m,w) = y_v^{(n+1)}(m,w),$$

$$x_v^{(n+1)}(m,w) = -\frac{\frac{1}{m}J_{w^{(n)}}\left(m, z^{(n+1)}\right) + \Lambda\left(z^{(n+1)}(w)\right)}{J_{w^{(n)}}\left(m, z^{(n+1)}\right) + \frac{1}{m}\Lambda\left(z^{(n+1)}(w)\right)} y_u^{(n+1)}(m,w). \qquad (10)$$

The following theorem concerns the properties of the successive approximations and their limits.

Theorem 2. *The sequences* $\left\{w^{(n)}(m,z)\right\}$, $\left\{z^{(n)}(m,w)\right\}$ *are compact in the sense of uniform convergence in the disk* $\overline{D}$. *Derivatives of the first order of the functions* $w^{(n)}$, $z^{(n)}$ *constitute the sequences compact in the space* $L_2(D)$ *in the sense of the weak convergence.*

Let

$$w(m,z) = \lim_{n\to\infty} w^{(n)}(m,z), \quad z(m,w) = \lim_{n\to\infty} z^{(n)}(m,w)$$

be the limit functions of the convergent sequences and $J_w(z)$, $J_z(w)$ *— their respective Jacobians. Then the functions* w, z *satisfy the following equations*

$$w_{\overline{z}}(m,z) = \frac{\Lambda(z) - J_w(z)}{J_w(z) + \Lambda(z)} \frac{1 - \frac{1}{m}}{1 + \frac{1}{m}} \overline{w}_z(m,z), \qquad (11)$$

$$z_{\overline{w}}(m,w) = \frac{1 - J_z(w)\Lambda\left(z(w)\right)}{1 + J_z(w)\Lambda\left(z(w)\right)} \frac{1 - \frac{1}{m}}{1 + \frac{1}{m}} z_w(m,w) \qquad (12)$$

almost everywhere in the disk D.

Proof. The sequences $\left\{w^{(n)}(m,z)\right\}$, $\left\{z^{(n)}(m,w)\right\}$ are the sequences of $K = K(m)$-quasiconformal mappings normalized by the correspondences of the three pairs of the boundary points. It means that they are compact in the sense of uniform convergence in the disk $\overline{D}$ [6–10].

Using the equations (8), (10) we get that the Dirichlet integrals of the functions $w^{(n)}$, $z^{(n)}$ are uniformly bounded in the space $L_2(D)$. From the theorem of Banach-Alaoglu [11] it follows that the sequences of the functions $w_z^{(n)}(m,z)$, $z_w^{(n)}(m,w)$, $w_{\overline{z}}^{(n)}(m,z)$, $z_{\overline{w}}^{(n)}(m,w)$ are compact in the space $L_2(D)$ in the sense of weak convergence.

Let us now prove that the functions w, z satisfy the equations (11), (12) respectively almost everywhere in D.

To this end we'll use the following quite evident lemmas.

Lemma 1. *Let*

$$J_\infty = \left\{z \in D | \lim_{n\to\infty} \sup J_{w^{(n)}}(z) = \infty\right\}$$

and $\mu(J_\infty)$ *— its Lebesgue measure. Then*

$$\mu(J_\infty) = 0.$$

Lemma 2. *There exists a sequence* $\{D_n\}$ *of the measurable sets* $D_n \subset D$ *such that*

$$\lim_{n\to\infty} \mu(D_n) = \pi$$

and the sequences

$$\left\{J_{w^{(n)}}\right\}, \quad \left\{w_{\overline{z}}^{(n)}(m,z)\right\}, \quad \left\{w_z^{(n)}\right\}$$

are uniformly bounded on each of the sets D_n.

Now in order to complete the proof of the theorem it is sufficient to show that the K-quasiconformal mapping $w = w(m, z)$ is the solution of the equation (11) on each of the set D_n and $z = z(m, w)$ is the solution of the equation (12) on the set $w(D_n)$.

It is clear that it is sufficient to this end to show that

$$\lim_{n\to\infty} \iint\limits_{D_n} w_z^{(k)}(m,z) J_{w^{(k)}}(z)\Phi(z)\mathrm{d}x\mathrm{d}y = \iint\limits_{D_n} w_z(m,z) J_w(z)\Phi(z)\mathrm{d}x\mathrm{d}y \qquad (13)$$

for any function $\Phi(z)$ continuous on the set D.

As we only know that the sequences $\{J_{w^{(k}}\}$, $\left\{w_z^{(k)}\right\}$ converge in the weak sense in the spaces $L^p(D_n)$, $p > 1$, the problem we are to solve seems not to be trivial [12]. But as the sequences $\{J_{w^{(n)}}\}$, $\left\{w_{\overline{z}}^{(n)}(m,z)\right\}$, $\left\{w_z^{(n)}\right\}$ are bounded on the sets D_n the equality (13) proves to be valid which leads us to the following lemma.

Lemma 3. *Let $\left\{w^{(n)}\right\}$ be a sequence of quasiconformal mappings converging almost everywhere to the mapping w. Let us suppose that the sequence $\{J_{w^{(n)}}\}$ converges weakly in the space $L^1(D)$ and the sequences $\left\{w_{\overline{z}}^{(n)}(m,z)\right\}$, $\left\{w_z^{(n)}\right\}$ converge weakly in the space $L^2(D)$. Let us suppose that sequences*

$$\{J_{w^{(n)}}\}, \quad \left\{w_{\overline{z}}^{(n)}(m,z)\right\}, \quad \left\{w_z^{(n)}\right\}$$

are uniformly bounded on each D_n. Then the equality (13) takes place for any function $\Phi(z)$ continuous on the set D.

From the lemma 3 we get now that almost everywhere on each set D_n the function satisfies the equation (11), i.e. it satisfies this equation almost everywhere in D.

In the same way we prove that the function $z(m, w)$ also satisfies the equation (12) almost everywhere in D.

The theorem is proved. □

3.2. Second stage

We have proved that there exists the sequence $\{w(m, z)\}$ of quasiconformal mappings satisfying equation (11). We can rewrite this equation in the following scalar form

$$\left(\frac{1}{m} + \Lambda^{-1}J_{w^{(n)}}\right) u_x(m,z) = \left(\frac{1}{m} - \Lambda^{-1}J_{w^{(n)}}\right) v_y(m,z),$$

$$\left(\frac{1}{m} + \Lambda^{-1}J_{w^{(n)}}\right) u_y(m,z) = -\left(\frac{1}{m} - \Lambda^{-1}J_{w^{(n)}}\right) v_x(m,z). \qquad (14)$$

Here the symbol $J_{w^{(n)}}$ denotes the Jacobian of the mapping $w(m, z) = u(m, z) + iv(m, z)$.

In what follows, we denote $w(m, z)$ as $w^{(m)}(z)$, $w^{(m)}(z) = u^{(m)}(z)+iv^{(m)}(z)$.

Theorem 3. *Let X be a surface of the class Ξ. Then there exists a sequence $\left\{w^{(m)}\right\}$ of the quasiconformal mappings $w^{(m)}(z)$, $w^{(m)}(z) = u^{(m)}(z) + iv^{(m)}(z)$, satisfying the equations (14) such that the sequence $\left\{u^{(m)}(z) + iv^{(m)2}\right\}$ is bounded in the space $W^{1,2}(D)$. The sequence $\left\{w^{(m)}\right\}$ converges uniformly on the disk D to the function $w(z) = u(z) + iv(z)$.*

The sequence $\left\{z^{(m)}\right\}$ *of the functions* $z^{(m)}$ *inverse to the functions* $w^{(m)}(z)$ *is bounded in the space* $W^{1,1}(D)$ *and compact in the sense of convergence in the space* $L^q(D)$, $1 \leqslant q < 2$.

Let D_c *be the set of pointwise convergence of the convergent subsequence of the sequence* $\left\{z^{(m)}\right\}$. *The function* $z = z(w) = \lim\limits_{n\to\infty} z^{(m)}$ *realizes injective mapping of the set* D_c *into the set* D. *The limit function* w *represents the inverse function to the function* $z = z(w)$ *on the set* $z(D_c)$ *and the function* $z = z(w)$ *defined over the set* D_c *is inverse to the function* $w = w(z)$ *there.*

The function $w = w(z)$ *satisfies the equations (11) almost everywhere in* D *and the function* $z = z(w)$ *satisfies the equation (12) almost everywhere in* D.

Besides, for almost all v, $-1 < v < 1$, *the lines* $\gamma_v = z(u, v)$, $-1 < u < 1$, *represent geodesics of the surface* X *covering the surface* X *up to the set of the null Hausdorff measure.*

Proof. Let us start with the following lemma.

Lemma 4. *The functions* $u^{(m)}$ *of the sequence* $\left\{u^{(m)}\right\}$ *are equicontinuous in* $\overline{D}$ *and uniformly bounded in the Sobolev space* $W^{1,2}(D)$.

Proof (Proof of the lemma 4). Firstly, let us note that from the equations (14) it follows that

$$u_x^{(m)2} + u_y^{(m)2} < \Lambda + \frac{1}{m} J_{w^{(m)}},$$
$$v_x^{(m)2} + v_y^{(m)2} < \Lambda + m J_{w^{(m)}}. \tag{15}$$

Let us consider the functions $\omega^{(m)}$,

$$\omega^{(m)} = u^{(m)} + \frac{i}{m} v^{(m)}.$$

The mappings $\omega^{(m)}$coincide with the mappings $w^{(m)}$ up to the affine mapping. It means that the mappings $\omega^{(m)}$ are topological ones. The inequality (15) means that the sequence of topological mappings $\omega^{(m)}$ is bounded in the space $W^{1,2}(D)$. This implies that the mappings $\omega^{(m)}$ are equicontinuous in the closed disk $\overline{D}$ [10].

The lemma is proved. □

We note that

$$\lim_{m\to\infty} \operatorname{Im} \omega^{(m)} = 0.$$

It means that we can judge only on the continuous properties of the functions $u^{(m)}$. As for the functions $v^{(m)}$ they stay for a while unknown to us. Nevertheless, it is easy to prove the following lemma.

Lemma 5. *The sequence* $\left\{v^{(m)2}\right\}$ *is uniformly bounded in the space* $W^{1,2}(D)$.

Using lemmas 4 and 5 we prove the following lemma.

Lemma 6. *The sequence* $\left\{w^{(m)}\right\}$ *is compact in the sense of the uniform convergence on the disk* D.

Let us now consider the inverse functions $z^{(m)}$ and prove the following lemma.

Lemma 7. *The sequence* $\left\{z^{(m)}\right\}$ *of the functions* $z^{(m)}$ *inverse to the functions* $w^{(m)}(z)$ *is bounded in the space* $W^{1,1}(D)$ *and is compact in the sense of convergence in the space* $L^q(D)$, $1 \leqslant q < 2$.

For almost all the lines, $\gamma_v = \{v = \text{const}\}$, $-1 \leqslant v \leqslant 1$, *the sequence* $\{z^{(m)}\}$ *is bounded in the space* $W^{1,2}(\gamma_v)$ *and it is compact in the sense of uniform convergence on such lines. The limit function* $z = z(w)$ *transforms the collection* V *of the lines* γ_v *of the uniform convergence of the sequence* $\{z^{(m)}\}$ *onto the set* $z(V)$ *covering almost all the disk* D.

Proof (Proof of the lemma 7). As the functions $z^{(m)}$ satisfy the equations (11) then the following property takes place for the derivatives of the functions $x^{(m)} = \operatorname{Re} z^{(m)}$, $y^{(m)} = \operatorname{Im} z^{(m)}$.

$$x_u^{(m)2} + y_u^{(m)2} + x_v^{(m)2} + y_v^{(m)2} \leqslant c\left(\frac{1}{\Lambda} + \Lambda J_z^2\right). \tag{16}$$

Using the inequality (16) we get that the sequence $\{z^{(m)}\}$ is bounded in the space $W^{1,1}(D)$. It means that the sequence $\{z^{(m)}\}$ is compact in the space $L^q(D)$, $1 \leqslant q < 2$ [13]. It is well-known [14] that it contains subsequence convergent almost everywhere in the disk D. As always, we denote the convergent subsequence as the sequence itself. This means that for almost all the lines γ_v, $-1 \leqslant v \leqslant 1$, the sequence $\{z^{(m)}\}$ converges almost everywhere on each of them. It is clear that the function w is inverse to the limit function $z = z(w)$ on the set $z(D_c)$. Moreover, the convergence of the bounded sequence $\{z^{(m)}\}$ is uniform on the lines γ_v. Really, the integrals

$$\int\limits_{\gamma_v} J_{z^{(m)}} \mathrm{d}u$$

are uniformly bounded for almost all v, $v \in (-1, 1)$.

From this property we easily get that for almost all v, $v \in (-1, 1)$ the integrals

$$\int\limits_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} \left[x_u^{(m)2} + y_u^{(m)2}\right] \mathrm{d}u$$

are uniformly bounded. It means that the functions $z^{(m)}$ of the sequence $\{z^{(m)}\}$ are equicontinuous on almost each of the lines γ_v, $-1 \leqslant v \leqslant 1$. The limit function z of convergent subsequence is absolutely continuous on almost all the lines γ_v, $-1 < v < 1$, as the function of the variable u and images of the lines γ_v do not intersect.

We are going to prove now that the image $z(V)$ cover the set D up to the set of the measure equal to zero.

Firstly, let us note that it is well-known [11] that there exists a function $J \in L^1(D)$ such that for each function Φ continuous and with compact support in D we have the following property

$$\iint\limits_D \left(J_{z^{(n)}}(w) - J(w)\right) \Phi(w) \mathrm{d}u \mathrm{d}v = 0.$$

Secondly, we note that the point z_0 such that $w(z_0) \in V$ belongs to $z(V)$.

Let us now suppose that there exists a set e of the positive measure such that $e^* = w(e) \subset V^c$. The complement V^c of the set V has null measure. Thus, we get that $\mu(e^*) = 0$. From this and weak convergence of $J_{z^{(n)}}$ we get a contradiction.

The lemma is proved. □

The following lemma whose proof is standard gives us the differential equations, which the functions $z = z(w)$, $w = w(z)$ satisfy.

Lemma 8. *The first order derivatives of the function $w = w(z) = u + iv$ exist, $u \in W^{1,2}(D)$, $v^2 \in W^{1,2}(D)$ and the function $w = w(z)$ satisfies the equation (6).*

The function $z = z(w)$ has generalized derivatives z_w, $z_{\overline{w}}$ belonging to the space $L^1(D)$, which satisfy the equation (7).

For the completion of the proof of the theorem it remains still to prove that almost all the lines $\{\nu = \text{const}\}$, $-1 < v < 1$, are geodesics.

For the solution of this problem we use standard approach [15].

We see now that all the properties of the functions $w(z)$, $z(w)$ were proved in the lemmas 4–8.

The theorem is proved. □

Summary

Using the theory of K-quasiconformal mappings we have constructed the solutions of non-linear Beltrami equations giving almost global half-geodesic parameterization of the twice differentiable surfaces with positive quadratic form.

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О существовании глобальной полугеодезической параметризации поверхностей

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В статье рассматривается задача о существовании глобальной полугеодезической параметризации поверхностей. Эта проблема хорошо известна и является до сих пор нерешённой в общем виде. Известно, что для дважды непрерывно дифференцируемых поверхностей эта проблема имеет локальное решение. Однако, пример параболоида вращения указывает на то, что невозможно, вообще говоря, использовать локальные сети для построения глобальной координатной сети, определяемой полугеодезической параметризацией. Для решения задачи авторы идут по пути, приводящему к построению изотермической параметризации для поверхностей с положительно определённой первой квадратичной формой. С этой целью они выводят дифференциальное уравнение, которому должно удовлетворять отображение реализующее нужную параметризацию.

В отличие от классического случая изотермической параметризации, новое уравнение представляет собой существенно нелинейное уравнение. Кроме того, эллиптическая система, определяемая новым уравнением, допускает вырождение в точках, в которых якобиан её решения обращается в ноль или бесконечность. При этом множество вырождения является заранее неизвестным. Неизвестна и скорость вырождения системы, которая существенно влияет на свойства неравномерно эллиптических систем.

Для преодоления указанных трудностей авторы видоизменяют постановку задачи: вместо семейства геодезических, покрывающих поверхность полностью, они ограничиваются семействами таких линий, которые покрывают её лишь с точностью до множества нулевой меры Хаусдорфа. С помощью теории K-квазиконформных отображений они строят негладкие отображения, являющиеся обобщёнными решениями нелинейного уравнения Бельтрами, которые, тем не менее, позволяют выделить нужное семейство геодезических. Построенная авторами параметризация даёт возможность исследовать неклассические равновесные формы жидких капель.

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