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THEORY OF THIN ELASTIC SHELLS

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An analysis of annular plate in curvilinear non-orthogonal coordinates with the help of equations of a shell theory

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Abstract

The complete system of equations of a linear theory of thin shells in curvilinear non-orthogonal coordinates proposed in the paper was taken as the basis of the investigation. Earlier, this system was used for static analysis of a long developable helicoid. In the article, this system is applied for the determination of stress-strain state of annular and circular plates under action of the external axisymmetric uniform load acting both in the plane of the plate and out-of-their plane. Presented results for annular plate given in the non-orthogonal coordinates expand a number of problems that can be solved analytically. They can be used as the first terms of series of expansion of displacements of degrees of the small parameter if a small parameter method is applied for examining a long tangential developable helicoid.

Keywords: arbitrary coordinates, shell theory, tangential developable helicoid, annular plate, equilibrium equations, axisymmetric load

Расчет кольцевой пластины в криволинейных неортогональных координатах с помощью уравнений теории оболочек

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Аннотация

В основу исследования положена полная система 20 уравнений в криволинейных неортогональных координатах линейной теории тонких оболочек, ранее использованная при статическом расчете длинного разверты-

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вающегося геликоида. В статье эта система применена для определения напряженно-деформированного состояния кольцевой и круглой пластин при внешней осесимметричной поверхностной нагрузке, действующей как в плоскости пластин, так и из их плоскости. Полученные результаты для кольцевой пластины в неортогональных координатах расширяют класс задач, которые теперь можно решить аналитически. Они могут быть использованы в качестве первых членов рядов разложения искомых перемещений в случае применения метода малого параметра применительно к длинному развертывающемуся геликоиду.

Ключевые слова: неортогональные координаты, теория оболочек, развертывающийся геликоид, кольцевая пластинка, уравнения равновесия, осесимметричная нагрузка

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1. Introduction

It is known, that the simplest equations of theory of thin shells are turned out for middle shell surfaces given in principle curvatures. But sometimes, it is very difficult to set a surface in principle curvatures and one must use governing equations of a theory of thin shells in curvilinear non-orthogonal coordinates. The complete system of equations in curvilinear non-orthogonal coordinates was proposed by A.L. Goldenveizer [1]. This system contains internal “pseudoforces”, “pseudomoments”, and Christoffel’s symbols. The system of equations, presented by Ya.M. Grigorenko and A.M. Timonin [2], is written in tensor form.

The system, proposed by the author, contains internal forces and moments usual for engineers and is free from Christoffel’s symbols [3]. Hereinafter, the equations presented in a paper [3] will be used.

2. An aim of investigation

Having the systems of the governing equations of a theory of thin shells, set by different scientists, it is desirable to use them for solution of problems of bending of plates and for solution of plane problems of their analysis, or to apply the governing equations of a theory of thin shells for solution of test examples of analysis of plane elements. This approach is illustrated by an example of reducing of general equations of a theory of thin shells in curvilinear arbitrary coordinates to the equations for analysis of tangential developable helicoid, and after for analysis of annular plates under action of axisymmetric uniform load of two types.

3. Methods of investigation

3.1. Governing equations of a theory of thin shells in curvilinear non-orthogonal coordinates

A system of the governing equations of a theory of thin shells in curvilinear non-orthogonal coordinates, proposed by the author, has the following form [3]:

– *six equilibrium equations*:

$$\frac{\partial}{\partial v}\left(AS_v\right)+\frac{N_u-N_v}{\sin\chi}\left(\frac{\partial B}{\partial u}-\frac{\partial A}{\partial v}\cos\chi\right)+\frac{\partial A}{\partial v}S_u+B\frac{\partial S_u}{\partial u}\cos\chi+B\frac{\partial N_u}{\partial u}\sin\chi-\frac{AB}{R_u}Q_u+ABX\sin\chi=0,$$

$$\frac{\partial}{\partial v}\left(AN_v\right)+\frac{S_u+S_v}{\sin\chi}\left(\frac{\partial B}{\partial u}-\frac{\partial A}{\partial v}\cos\chi\right)-\frac{\partial A}{\partial v}N_u+B\frac{\partial S_u}{\partial u}\sin\chi-B\frac{\partial N_u}{\partial u}\cos\chi-\frac{AB}{\sin\chi}\left(\frac{Q_v}{R_v}-\frac{Q_u}{R_u}\cos\chi\right)+ABY\sin\chi=0,$$

$$\frac{N_u}{R_u\sin\chi}+\frac{N_v}{R_v\sin\chi}+\frac{1}{AB}\left[\frac{\partial}{\partial u}\left(BQ_u\right)+\frac{\partial}{\partial v}\left(AQ_v\right)\right]-Z\sin\chi=0,$$

$$-\frac{\partial}{\partial v}(AM_v)+\frac{M_{uv}+M_{vu}}{\sin\chi}\left(\frac{\partial B}{\partial u}-\frac{\partial A}{\partial v}\cos\chi\right)+\frac{\partial A}{\partial v}M_u+B\frac{\partial M_u}{\partial u}\cos\chi+B\frac{\partial M_{uv}}{\partial u}+\sin\chi+ABQ_v\sin\chi=0,$$

$$\frac{\partial}{\partial v}\left(AM_{vu}\right)+\frac{M_v-M_u}{\sin\chi}\left(\frac{\partial B}{\partial u}-\frac{\partial A}{\partial v}\cos\chi\right)+\frac{\partial A}{\partial v}M_{uv}-B\frac{\partial M_u}{\partial u}\sin\chi+B\frac{\partial M_{uv}}{\partial u}\cos\chi+AB(Q_u+Q_v\cos\chi)=0,$$

$$(S_u - S_v)\sin\chi + (N_v - N_u)\cos\chi + \frac{M_{uv}}{R_u \sin\chi} - \frac{M_{vu}}{R_v \sin\chi} = 0, \tag{1}$$

where *X, Y, Z* – external uniform surface load in the direction of mobile orthogonal axes *x, y, z,* and the *x*-axis coincides with the tangent to the *u*-coordinate line;

– *six geometrical equations* suggested by A.L. Goldenveizer and presented in a monography [1], the first three of which after submission of Christoffel's symbols in them can be expressed as [3]

$$\varepsilon_u = \frac{1}{A}\left[\frac{\partial u_u}{\partial u} + \frac{\partial A}{\partial v}\left(\frac{u_v}{B}\right) + B\cos\chi\frac{\partial}{\partial u}\left(\frac{u_v}{B}\right)\right] - \frac{u_z}{R_u^{/}},$$

$$\varepsilon_v = \frac{1}{B}\left[\frac{\partial u_v}{\partial v} + \frac{\partial B}{\partial u}\left(\frac{u_u}{A}\right) + A\cos\chi\frac{\partial}{\partial v}\left(\frac{u_u}{A}\right)\right] - \frac{u_z}{R_v^{/}},$$

$$\varepsilon_{uv} = \sin\chi\left[\frac{B}{A}\frac{\partial}{\partial u}\left(\frac{u_v}{B}\right) + \frac{A}{B}\frac{\partial}{\partial v}\left(\frac{u_u}{A}\right)\right] - \frac{1}{A}\frac{\partial\chi}{\partial u}u_u - \frac{1}{B}\frac{\partial\chi}{\partial v}u_v + \frac{u_z}{\sin\chi}\left(\frac{2}{R_{12}} + \frac{\cos\chi}{R_u^{/}} + \frac{\cos\chi}{R_v^{/}}\right). \tag{2}$$

Remaining three equations for the determination of change of curvatures κ_u and κ_v and torsion κ_{uv} have rather complex form, for example, for shallow shells

$$\kappa_u = \frac{1}{A}\left[\frac{\partial}{\partial u}\left(\frac{1}{A\sin\chi}\frac{\partial u_z}{\partial u}\right) - \frac{1}{B}\frac{\partial}{\partial u}\left(\frac{\cos\chi}{\sin\chi}\frac{\partial u_z}{\partial v}\right) + \frac{1}{B^2\sin\chi}\frac{\partial A}{\partial v}\frac{\partial u_z}{\partial v}\right],$$

$$\kappa_v = \frac{1}{B}\left[\frac{\partial}{\partial v}\left(\frac{1}{B\sin\chi}\frac{\partial u_z}{\partial v}\right) - \frac{1}{A}\frac{\partial}{\partial v}\left(\frac{\cos\chi}{\sin\chi}\frac{\partial u_z}{\partial u}\right) + \frac{1}{A^2\sin\chi}\frac{\partial B}{\partial u}\frac{\partial u_z}{\partial u}\right],$$

$$\kappa_{uv} = \frac{1}{AB\sin\chi}\left[\frac{B^2}{2}\frac{\partial}{\partial u}\left(\frac{1}{B^2}\frac{\partial u_z}{\partial v}\right) + \frac{A^2}{2}\frac{\partial}{\partial v}\left(\frac{1}{A^2}\frac{\partial u_z}{\partial u}\right) - \cos^2\chi\frac{\partial^2 u_z}{\partial u\partial v} + \frac{\cos\chi}{A}\frac{\partial B}{\partial u}\frac{\partial u_z}{\partial u} + \frac{\cos\chi}{B}\frac{\partial A}{\partial v}\frac{\partial u_z}{\partial v}\right]. \tag{3}$$

Eight physical equations connecting internal forces N_u, N_v, S_u, S_v, Q_u, Q_v and moments M_u, M_v, M_{uv}, M_{vu} and components of tangential and bending deformations ε_u, ε_v, ε_{uv}, κ_u, κ_v, κ_{uv} between themselves can be written as [3]

$$N_v = \frac{Eh}{1-\nu^2}(\varepsilon_v - \varepsilon_{uv}\mathrm{ctg}\chi + \nu\varepsilon_u),\ N_u = \frac{Eh}{1-\nu^2}(\varepsilon_u - \varepsilon_{uv}\mathrm{ctg}\chi + \nu\varepsilon_v),$$

$$S_v = \frac{1-\nu}{2}C\left[\varepsilon_{uv} + (\varepsilon_v - \varepsilon_u)\mathrm{ctg}\chi\right],\ S_u = \frac{1-\nu}{2}C\left[\varepsilon_{uv} + (\varepsilon_u - \varepsilon_v)\mathrm{ctg}\chi\right],$$

$$M_{vu} = \frac{Eh^3}{12(1+\nu)}\left(\kappa_{vu} - \kappa_u\cos\chi\right),\ M_{uv} = \frac{Eh^3}{12(1+\nu)}\left(\kappa_{vu} - \kappa_v\cos\chi\right),$$

$$M_v = -\frac{Eh^3}{12(1-\nu^2)}\left[\frac{\kappa_u + \kappa_v}{\sin\chi} - (1-\nu)(\kappa_u\sin\chi + \kappa_{uv}\mathrm{ctg}\chi)\right],$$

$$M_u = -\frac{Eh^3}{12(1-\nu^2)}\left[\frac{\kappa_u + \kappa_v}{\sin\chi} - (1-\nu)(\kappa_v\sin\chi + \kappa_{uv}\mathrm{ctg}\chi)\right], \tag{4}$$

where $\chi \neq \pi/2$ is the angle between the coordinate lines *u* and *v*; ν = Poisson's ratio; *h* = thickness of shell.

A vector of displacements can be written as

$$\boldsymbol{U} = u_u\frac{\boldsymbol{r}_u}{A} + u_v\frac{\boldsymbol{r}_v}{B} - u_z\boldsymbol{n}.$$

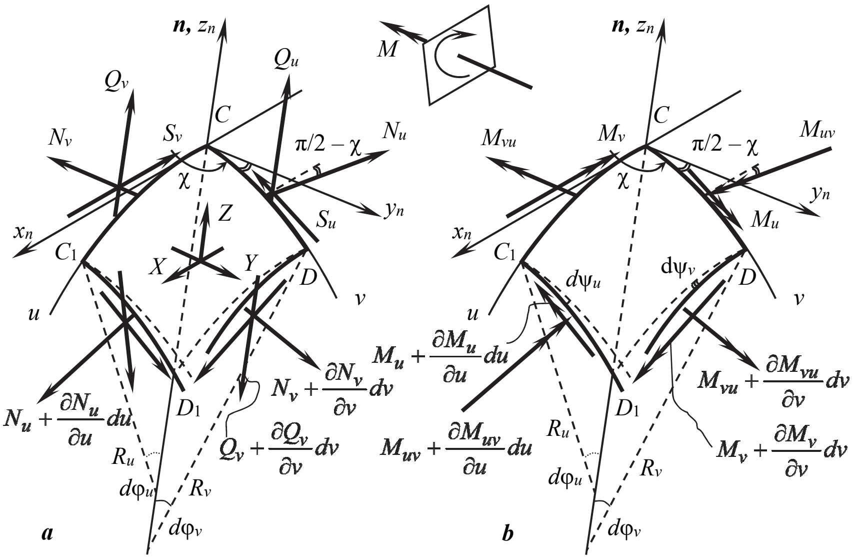

Figure. The forces (*a*) and moments (*b*), per unit length, needed for equilibrium

The forces and moments, per unit length, needed for equilibrium are shown in Figure and are positive as shown.

3.2. Developable helicoid

Parametric equations of evolvent (developable) helicoid can be written as

$$x = x(u,v) = a\cos v - au\sin v / m,$$

$$y = y(u,v) = a\sin v + au\cos v / m,$$

$$z = z(u,v) = bv + bu / m, \tag{5}$$

where $m = \sqrt{a^2 + b^2}$; b is the lead of a helix u = 0 (cuspidal edge); v is an angle measured from an axis Ox; a is a radius of a cylinder on which the helical cuspidal edge is lying.

In this case, coefficients of the fundamental forms of the surface (5) and its principal curvatures (k_u; k_v) can be written as

$$A = 1;\quad F = m;\quad B^2 = m^2 + u^2a^2/m^2;\quad N = uab/m^2;\quad L = M = 0;\quad k_u = 0;\quad k_v = N/B^2, \tag{6}$$

and also

$$\cos\chi = \frac{F}{AB} = \frac{m}{B},\quad \sin\chi = \frac{ua}{mB}, \frac{\partial B}{\partial u} = \frac{ua^2}{m^2 B}. \tag{7}$$

The u-coordinate lines coincide with the straight generatrixes of the helicoid but the v-lines are co-axis helixes. The formulae (6) show that conjugated ($M = 0$) non-orthogonal ($F \neq 0$) system of curvilinear coordinates is used.

For a developable helicoid (5) with coefficients of the fundamental forms (6), the equations (1)–(3) become simpler. Substituting the geometrical equations (2), (3) into the formulae of Hooke's law (4) but the results into the equilibrium equations (1) one can obtain a system of three differential equations in partial derivatives of the 8th order. This was made in a monograph [3]. But this system of equations was not solved analytically.

A system of three differential equations in partial derivatives of the 8th order, presented in a work [4], was reduced to a system of three ordinary differential equations of the 8th order for a long developable helicoid. It was assumed that all components of stress-strain state of a long developable helicoid depend only on the u-coordinate, i. e. all derivatives with respect to parameter v are equal to zero. In this case, a problem of determination of components of stress-strain state yields to analytical solution. With the help of small parameter method, it was solved in works [4; 5]. With the help of asymptotic semianalytical method, it was solved in a work [6]. Analogous approach was used in a paper [7] for analysis of a long right helicoid. Two methods of analysis of thin elastic open helicoidal shells were used in a manuscript [8] where the equations of A.L. Goldenveizer [1] and the equations (1)–(4) were applied. Now five types of helicoidal shells are known [9]. All these shells can be analyzed with the help of presented equations (1)–(4) [10; 11].

3.3. Governing equations for thin annular plates in curvilinear non-orthogonal coordinates

For annular plates, we have $b = 0$, therefore, formulae (1)–(7) are simplified and become

$$m = a;\quad A = 1;\quad F = a;\quad B^2 = a^2 + u^2;$$

$$N = L = M = 0;\quad k_u = 0;\quad k_v = 0;$$

$$\cos\chi = \frac{a}{B},\quad \sin\chi = \frac{u}{B}, \frac{\partial B}{\partial u} = \frac{u}{B}, \frac{\partial \chi}{\partial u} = \frac{a}{B^2}. \tag{8}$$

Substituting the values (8) into formulae (1)–(7) gives the possibility easily to obtain corresponding governing equations for annular plate or its fragment subjected to arbitrary uniform surface load or to linear load along contour of the plate acting in the plane of the plate.

One can obtain a system of three differential equations in partial derivatives in displacements and reduce them to a system of two differential equations in displacements of the same order. But to solve analytically the obtained systems it is not possible yet. Of course, this problem can be solved with the help of FEM [12].

Assume that an annular plate is subjected to an axisymmetric load. Then, all parameters of stress-strain state of the plate will not depend on the v-coordinate, i. e.

$$\frac{\partial^i \ldots}{\partial v^i} = 0.$$

Hence, the system of equations (1)–(4) becomes:

– *equilibrium equations*:

$$\frac{d}{du}(uN_u + aS_u) - N_v + uX = 0,$$

$$\frac{d}{du}(uS_u - aN_u) + S_v + uY = 0,$$

$$\frac{d}{du}(BQ_u) - uZ = 0,$$

$$a\frac{dM_u}{du} + M_{vu} + uQ_v = 0,$$

$$-\frac{d(uM_u)}{du} + M_v + aQ_v + BQ_u = 0,$$

$$u(S_u - S_v) + a(N_v - N_u) = 0; \tag{9}$$

– *geometrical equations:*

$$\varepsilon_u = \frac{d\theta}{du},\quad \varepsilon_v = \frac{u}{B^2}u_u,$$

$$\varepsilon_{uv} = u\frac{d\psi}{du} - \frac{a}{B^2}u_u,$$

$$\kappa_u = \frac{d}{du}\left(\frac{B}{u}\frac{du_z}{du}\right),\quad \kappa_v = \frac{1}{B}\frac{du_z}{du},$$

$$\kappa_{uv} = \frac{a}{B^2}\frac{du_z}{du}; \tag{10}$$

– *physical equations:*

$$N_v = C\left[-\frac{a}{u}\frac{d}{du}(u\psi) + \frac{\theta}{u} + \nu\frac{d\theta}{du}\right],$$

$$N_u = C\left[\frac{du_u}{du} + \frac{1}{u}u_u - \frac{1-\nu}{B^2}u\cdot u_u\right],$$

$$S_v = \frac{1-\nu}{2u}C\left(u^2\frac{d\psi}{du} - a\frac{d\theta}{du}\right),$$

$$S_u = \frac{1-\nu}{2}C\left(u\frac{d\psi}{du} + \frac{a}{u}\frac{d\theta}{du} - \frac{2a}{B^2}u_u\right),$$

$$M_{vu} = -(1-\nu)aD\frac{d}{du}\left(\frac{1}{u}\frac{du_z}{du}\right), \quad M_{uv} = 0,$$

$$M_v = -\frac{D}{u}\left[\frac{d}{du}\left(\frac{B^2}{u}\frac{du_z}{du}\right) - (1-\nu)u\frac{d^2u_z}{du^2}\right],$$

$$M_u = -\frac{D}{u}\left[\frac{d}{du}\left(\frac{B^2}{u}\frac{du_z}{du}\right) - (1-\nu)\frac{du_z}{du}\right], \tag{11}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}, \quad C = \frac{Eh}{1-\nu^2}.$$

New notations were introduced in formulae (10) and (11):

$$\theta = u_u + \frac{a}{B}u_v, \quad \psi = \frac{u_v}{B}. \tag{12}$$

4. Results

4.1. Applied load is perpendicular to the middle surface of the plate

Assume that an annular plate subjected to constant surface load Z = const. It means that applied load is perpendicular to the middle surface of the plate. This assumption eliminates the need to consider in-plane membrane forces that are not considered in the classical theory of bending of thin plates. So, integrating the third equation of equilibrium (9), we determine

$$BQ_u = \frac{u^2}{2}Z + \widetilde{C_1}, \tag{13}$$

where $\widetilde{C_1}$ is the unknown constant of the integration.

Substituting the values of the shear force Q_v, obtained from the fourth equation of equilibrium (9), into the fifth equation of equilibrium, gives

$$BQ_u = \frac{B^2}{u^2}\frac{d(uM_u)}{du} - M_v - \frac{a^2}{u^2}M_u + \frac{a}{u}M_{vu}.$$

Taking into account expressions (11), the obtained equation can be written as

$$BQ_u = -D\frac{B^2}{u}\frac{d}{du}\left[\frac{1}{u}\frac{d}{du}\left(\frac{B^2}{u}\frac{du_z}{du}\right)\right],$$

or with the help of an equality (13) we get

$$-D\frac{B^2}{u}\frac{d}{du}\left[\frac{1}{u}\frac{d}{du}\left(\frac{B^2}{u}\frac{du_z}{du}\right)\right] = \frac{u^2}{2}Z + \widetilde{C_1}.$$

Integration of the last expression gives

$$u_z = C_1 B^2 \ln B^2 + C_2 \ln B^2 + C_3 u^2 + C_4 - \frac{Z}{64D} u^4, \tag{14}$$

where C_1, C_2, C_3, C_4 are the unknown constants of the integration, that can be determined after satisfaction of boundary conditions.

Substituting a value of displacement (14) in the expression for the determination of BQ_u gives

$$BQ_u = -8DC_1 + \frac{u^2}{2} Z,$$

$$\text{i. e. } \breve{C_1} = -8DC_1.$$

At last, we can find

$$Q_v = -\frac{a}{u}\frac{dM_u}{du} - \frac{1}{u} M_{vu} = \frac{8a}{B^2} DC_1 - \frac{a}{2} Z.$$

4.2. Applied load is in the plane of the plate

Examine loading annular or circular plate ($a = 0$) by axisymmetric load acting in the plane of the plate. In this case, we use the first two equations of equilibrium (9). After substitution the first fourth physical equations in them and after some transformations, equations (9) take the following form:

$$\frac{d}{du}\left\{\frac{1}{u}\frac{d}{du}[u(\theta - a\psi)]\right\} = \frac{d}{du}\left\{\frac{1}{u}\frac{d}{du}[u \cdot u_u]\right\} = \frac{u}{CB^2}(aY - uX),$$

$$\frac{d}{du}\left(u^3 \frac{d\psi}{du}\right) + a\frac{d}{du}\left(u\frac{d\theta}{du}\right) - 2a\frac{d\theta}{du} = -\frac{2u^3}{(1-\nu)CB^2}(aX + uY).$$

Integration of two last expressions gives

$$\theta - a\psi = u_u = \frac{1}{u}\int \left[u \int \frac{u}{CB^2}(aY - uX) du\right] du + \frac{u}{2} D_1 + \frac{1}{u} D_2,$$

$$\psi + \frac{a}{u^2}\theta = -\int \frac{1}{u^3} \int \frac{2u^3 (aX + uY)}{CB^2 (1-\nu)} du du + \frac{D_3}{2u^2} + D_4, \tag{15}$$

where D_1, D_2, D_3, D_4 are the unknown constants of the integration.

The last two equations give the possibility to find parameters of deflections (12). Substituting these parameters into the physical equations (11) one can find the internal forces and moments in an annular or circular plate subjected to an external uniform axisymmetric load acting in the plane of the plate.

Assume that $X = Y = 0$ then from the last two expressions we determine

$$\theta = \frac{1}{B^2}\left(\frac{u^3}{2} D_1 + u D_2 + \frac{a}{2} D_3 + a u^2 D_4\right),$$

$$\psi = \frac{u_v}{B} = \frac{1}{B^2}\left(-\frac{au}{2} D_1 - \frac{a}{u} D_2 + \frac{1}{2} D_3 + u^2 D_4\right),$$

$$u_u = \frac{u}{2} D_1 + \frac{1}{u} D_2.$$

The last expression can be applied if uniform loads *X, Y* are absent, but displacements given in advance or external moments or forces given in advance are known on the plate edges.

In a paper [13], the analogous axisymmetric problem was solved by using finite difference method.

5. Conclusion

If one assumes that a radius of the inner edge of the annular plate $a = 0$ then $B = u$, $\chi = \pi/2$ and the well-known equations for circular plate in the polar coordinates would be obtained [14].

It shows that governing equations of a shell theory (1)–(4) assumed as a basis of presented investigation are correct. The results obtained for annular plate in non-orthogonal coordinates widen a class of problems which now can be solved analytically. Examining a long tangential developable helicoid with the help of a small parameter method [15], it is possible to use the solutions (14) and (15) as the first terms of series of expansion of displacements of degrees of the small parameter. Using equations (8)–(15) it is possibly to obtain analytical solutions for plates with different axisymmetric static load and support conditions presented in a paper [16].

The additional information on the application of annular and circular plates and on numerical and analytical analysis of thin and thick plates is given in manuscripts [17–20].

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