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The asymptotic solution of a singularly perturbed Cauchy problem for Fokker–Planck equation

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The asymptotic method is a very attractive area of applied mathematics. There are many modern research directions which use a small parameter such as statistical mechanics, chemical reaction theory and so on. The application of the Fokker–Planck equation (FPE) with a small parameter is the most popular because this equation is the parabolic partial differential equations and the solutions of FPE give the probability density function.

In this paper we investigate the singularly perturbed Cauchy problem for symmetric linear system of parabolic partial differential equations with a small parameter. We assume that this system is the Tikhonov non-homogeneous system with constant coefficients. The paper aims to consider this Cauchy problem, apply the asymptotic method and construct expansions of the solutions in the form of two-type decomposition. This decomposition has regular and border-layer parts. The main result of this paper is a justification of an asymptotic expansion for the solutions of this Cauchy problem. Our method can be applied in a wide variety of cases for singularly perturbed Cauchy problems of Fokker–Planck equations.

Key words and phrases: asymptotic analysis, singularly perturbed differential equation, Cauchy problem, Fokker–Planck equation

1. Introduction

It is well known that the differential operator, which is applied in the theory of measure, has such form:

$$L = a^{ij}\partial_{x_i}\partial_{x_j} + b^i\partial_{x_i}, \quad 1 \leqslant i, j \leqslant d, \quad d \in N.$$

The solution of the equation $L^*\mu = 0$ is Borel measures on an open set $\Omega \in \mathbf{R}^d$ and there is the relation

$$\int_\Omega Lf d\mu = 0, \quad \forall f \in C_0^\infty(\Omega).$$

If the measure μ has a density ρ, then ρ is conjugate solution of the equation

$$\partial_{x_i}\partial_{x_j}a^{ij}\rho(x) - \partial_{x_i}b^i\rho(x) = 0, \quad x \in \Omega.$$

Similarly, we can consider parabolic operators in the form

$$P = \partial_t - \partial_{x_i}\partial_{x_j}a^{ij} + \partial_{x_i}b^i,$$

and there are appropriate parabolic equations $P^*\mu = 0$ for finding measures μ on $\mathbf{R}^n \times [0, T]$. The equations for the study of density have the form of Fokker–Planck equation (FPE)

$$\partial_t\rho(x,t) - \partial_{x_i}\partial_{x_j}a^{ij}(x,t)\rho(x,t) + \partial_{x_i}b^i(x,t)\rho(x,t) = 0.$$

FPE equation uses for analysis a macroscopic process but for a small subsystem.

We can formulate the singularly perturbed Cauchy problem for FPE in the form:

$$\varepsilon\partial_t\rho(x,t,\varepsilon) - \partial_{x_i}\partial_{x_j}a^{ij}(x,t)\rho(x,t,\varepsilon) + \partial_{x_i}b^i(x,t)\rho(x,t,\varepsilon) = 0,$$

$$\rho(x,0,\varepsilon) = \rho_0(x), \quad x \in \Omega, \quad \forall\rho_0(x) \in C_0^\infty(\Omega),$$

where $\varepsilon > 0$ is a small parameter.

If we assume $\varepsilon = 0$, we can get a degenerate Cauchy problem in the following form:

$$\partial_{x_i}\partial_{x_j}a^{ij}(x,t)\bar{\rho}(x,t) - \partial_{x_i}b^i(x,t)\bar{\rho}(x,t) = 0,$$

$$\bar{\rho}(x,0) = \rho_0(x), \quad x \in \Omega, \quad \forall\rho_0(x) \in C_0^\infty(\Omega),$$

where solutions $\bar{\rho}(x,t)$ are solutions of the degenerate problem and $\bar{\rho}$ may differ from solutions $\rho(x,t)$ significantly.

A large number of methods have been developed for the analytical and numerical study of FPE solutions [1]–[8]. Hyung Ju Hwang and Jinoh Kim [9], [10] study the initial–boundary value problem for the Vlasov–Poisson–Fokker–Planck equations in an interval with absorbing boundary conditions. They introduce the Deep Neural Network (DNN) approximated solutions to the kinetic Fokker–Planck equation in a bounded interval and study the large-time asymptotic behavior of the solutions and other physically relevant macroscopic quantities. Shu-Nan Li and Bing-Yang Cao [11] obtained solutions based on the fractional Fokker–Planck equation (FFPE) with a generic time- and length-dependence of an "effective thermal conductivity" (κ_{eff}), namely, $\kappa_{\text{eff}}L\alpha$ with L being the system length. They formulate the effective thermal conductivity in terms of entropy generation, which does not rely on the local-equilibrium hypothesis. Hrishikesh Patel and Bernie D. Shizgal [12] compare the Kappa distribution of space plasmas modelled with a particular Fokker–Planck equation for a two component system with the linear Fokker–Planck equation that has been used to study the Student t-distribution. Lucas Philip and Bernie D. Shizgal [13] consider the one-dimensional bistable Fokker–Planck equation with specific drift and diffusion coefficients so as to model protein folding. Yunfei Su and Lei Yao

[14] study the hydrodynamic limit for the inhomogeneous incompressible Fokker–Planck equations.

The development of the asymptotic analysis of singularly perturbed differential equations and systems of differential equations was made by A. N. Tikhonov [15], M. I. Vishik and L. A. Lyusternik [16], A. B. Vasil'eva [17], S. A. Lomov [18], V. A. Trenogin [19], J. L. Lions [20] and other researchers during the second half of the 20th century. There is a large number of recent works. O. Hawamdeh and A. Perjan [21] study an asymptotic expansions for linear symmetric hyperbolic systems with small parameter. Using the boundary layer functions method of Lyusternik–Vishik, A. Perjan [22] obtains the asymptotic expansions of the solutions to the Cauchy problem for the linear symmetric hyperbolic system as the small parameter $\varepsilon \to 0$. A. N. Gorban [23] investigates a model reduction in chemical dynamics with slow invariant manifolds and singular perturbations. Bor-Yann Chen, Liying Wu and Junming Hong [24] consider singular limits of reaction diffusion equations and geometric flows with discontinuous velocity.

In this paper we apply the results of the paper [21] and investigate the Cauchy problem for the singularly perturbed Tikhonov-type symmetric system of non-homogeneous constant coefficients linear parabolic partial differential equations (LPPDE system) with a small parameter. We use the asymptotic method for this Cauchy problem and construct expansions of solutions in the form of decomposition, which has regular and border-layer parts. The main result of this paper is a proof of a justification theorem of an asymptotic expansion for this Cauchy problem. Our method can be applied in a wide variety of cases for singularly perturbed Cauchy problems of Fokker–Planck equations.

2. Singularly perturbed Cauchy problem for LPPDE system

We consider the following singularly perturbed Cauchy problem (P_ϵ),

$$P_\epsilon u(x,t,\varepsilon) = f(x,t), \quad x \in \mathbf{R}^d, \quad t \geqslant 0, \tag{1}$$

$$u(x,0,\varepsilon) = u_0(x), \quad x \in \mathbf{R}^d, \tag{2}$$

where $\varepsilon > 0$ is a small parameter.

Thus, $P_\epsilon = P_0 + \varepsilon P_1$ is a parabolic operator, where $P_i = A_i \partial_t + B_i(\partial_x) + D_i$, $i = 0, 1$,

$$B_i(\partial_x) = \sum_{p=1}^{d} B_{ip}\partial_{x_p} - \sum_{p,q=1}^{d} C_{ipq}\partial_{x_p}\partial_{x_q},$$

$B_{ip} = (b_{st}^{ip})_{s,t=1}^{n}$, $C_{ipq} = (c_{st}^{ipq})_{s,t=1}^{n}$, $D_i = (d_{st}^{i})_{s,t=1}^{n}$ are real constants of symmetric $n \times n$ matrices and $b_{st}^{ip} \geqslant 0$, $c_{st}^{ipq} \geqslant 0$, $d_{st}^{i} \geqslant 0$ ($\forall s,t = 1,\dots,n$), $d \geqslant 1$, $u(x,0,\varepsilon) : \mathbf{R}^d \times [0,\infty) \times (0,\infty) \to \mathbf{R}^n$, $f(x,t) : \mathbf{R}^d \times [0,\infty) \to \mathbf{R}^n$, $f(x,t) \in C^1$,

$$A_0 = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-m} \end{pmatrix}, \quad 0 \leqslant m \leqslant n,$$

where I_k is an identity matrix and

$$A = A_0 + \varepsilon A_1, \quad B(\partial_x) = B_0(\partial_x) + \varepsilon B_1(\partial_x), \quad D = D_0 + \varepsilon D_1,$$

$$L_i(\partial_x) = B_i(\partial_x) + D_i, \quad i = 0, 1, \quad \partial_x = (\partial/\partial_{x_1}, \dots, \partial/\partial_{x_d}).$$

The special forms of matrices A_0 and A_1 determine the natural representations of matrices B_i, D_i by blocks in the forms:

$$B_i(\partial_x) = \begin{pmatrix} B_{i1}(\partial_x) & B_{i2}(\partial_x) \\ B^*_{i2}(\partial_x) & B_{i3}(\partial_x) \end{pmatrix}, \quad D_i = \begin{pmatrix} D_{i1} & D_{i2} \\ D^*_{i2} & D_{i3} \end{pmatrix}, \quad i = 0, 1,$$

where $B_{i1}(\partial_x)$, $D_{i1} \in M^{m\times m}(\mathbf{R})$, $B_{i2}(\partial_x)$, $D_{i2} \in M^{m\times(n-m)}(\mathbf{R})$, $B_{i3}(\partial_x)$, $D_{i3} \in M^{(n-m)\times(n-m)}(\mathbf{R})$, and $*$ means transposition, and

$$B_{ij}(\partial_x) = \sum_{p=1}^{d} B^{ij}_p \partial_{x_p} - \sum_{p,q=1}^{d} C^{ij}_{pq} \partial_{x_p} \partial_{x_q}, \quad i = 0, 1,\ j = 1, 2, 3,$$

$$B^{01}_p = (b^{0p}_{st})_{s,t=\overline{1,m}}, \quad C^{01}_{pq} = (c^{0pq}_{st})_{s,t=\overline{1,m}},$$

$$B^{02}_p = (b^{0p}_{st})_{s=\overline{1,m},t=\overline{m+1,n}}, \quad C^{02}_{pq} = (c^{0pq}_{st})_{s=\overline{1,m},t=\overline{m+1,n}},$$

$$B^{03}_p = (b^{0p}_{st})_{s,t=\overline{m+1,n}}, \quad C^{02}_{pq} = (c^{0pq}_{st})_{s,t=\overline{m+1,n}}.$$

The aim of our work is to construct the asymptotic solution $u(\varepsilon, x, t)$ for (P_ε) with a small parameter $\varepsilon \to 0$.

Thus, the investigation of the solution $u(\varepsilon, x, t)$ depends on the structure of the operator P_ε. The norm, which determines the convergence of the perturbed system solution, is also very important.

We denote the usual Sobolev spaces by H^s with the scalar product in the form:

$$(u, v)_s = \int_{\mathbf{R}^d} (1 + \xi^2)^s \hat{u}(\xi) \bar{\hat{v}}(\xi)\, d\xi,$$

where $s \in \mathbf{R}$, $\hat{u}(\xi) = F[u]$ ($\xi \in \mathbf{R}^d$) and $F^{-1}[u]$ are the direct and the inverse Fourier transforms of the function u in S'. Let $H^s_d = (H^s)^d$ be a notation of the Hilbert space, which is associated with the scalar product

$$(f_1, f_2)_{s,d} = \sum_{j=1}^{d} (f_{1j}, f_{2j})_s, \quad f_i = (f_{i1}, \dots, f_{id}), \quad i = 1, 2,$$

and with the norm $\|\cdot\|_{s,d}$, which is generated by this scalar product.

Let $D'((a,b),X)$ be a space of vectorial distributions on (a,b) with values in Banach space X. We can set

$$W^{k,p}(a,b;X) = \{u \in D'((a,b);X);\ u^{(j)} \in L^p(a,b;X),\ j = 0,1,\dots,k\},$$

for $k \in \mathbb{N}^*$ and $1 \leqslant p \leqslant \infty$, where $u^{(j)}$ is the distributional derivative of order j and $W^{0,p}(a,b;X) = L^p(a,b;X)$ for $k=0$.

We denote operator $L_{ij}(\partial_x)$ in the form:

$$L_{ij}(\partial_x) = B_{ij}(\partial_x) + D_{ij},$$

and

$$F = \mathrm{col}(f,g), \quad U_0 = \mathrm{col}(u_0,u_1),$$

where $f, u_0 \in M^{m\times 1}(\mathbf{R})$, $g, u_1 \in M^{(n-m)\times 1}(\mathbf{R})$.

We assume that

H1: B_{ip}, C_{ipq}, D_i, $i = 0,1$, $p,q = \overline{1,d}$ are real symmetric matrices;

H2: $(D\zeta,\zeta)_{\mathbf{R}^n} \geqslant (D_{03}\eta,\eta)_{\mathbf{R}^{n-m}} \geqslant q_0|\eta|^2$, with $q_0 > 0$; for all $\zeta \in \mathbf{R}^n$ and $\eta \in \mathbf{R}^{n-m}$.

Thus, the operator (P_ε) is a symmetric parabolic system (**H1**) and the operator (P_0) is an elliptic-parabolic system in case: *det* $B_{03} \neq 0$ and $B_{02} = 0$.

3. Formal asymptotic expansions of the singularly perturbed Cauchy problem (P_ϵ)

We construct the formal asymptotic expansions of the solutions $u(\varepsilon,x,t)$ for the Cauchy problem (P_ε) on the positive powers of the small parameter ε in this section.

We can use the following asymptotic expansion of the solution $u(\varepsilon,x,t)$ for the problem (P_ε) according to the method of Lyusternik–Vishik [16]:

$$u(\varepsilon,x,t) = V(x,t,\varepsilon) + Z(x,\tau) = \sum_{k=0}^{N} \varepsilon^k (V_k(x,t) + Z_k(x,\tau)) + R_N(\varepsilon,x,t), \quad (3)$$

where $\tau = t/\varepsilon$, and $Z(x,\tau) = Z_0(x,\tau) + \dots + \varepsilon^N Z_N(x,\tau)$ is the boundary layer function, which describes the singular behavior of the solution $u(\varepsilon,x,t)$ within a neighborhood of the set $\{(x,0),\ x \in \mathbf{R}^d\}$, which is the boundary layer.

The function $V(x,t) = V_0(x,t) + \dots + \varepsilon^N V_N(x,t)$ is the regular part of expansion (3).

We assume that the function $Z(x,\tau)$ is small for large τ, i.e. $Z \to 0$ as $\tau \to \infty$. There is the solutions behavior $u(\varepsilon,x,t) \not\to u(0,x,t)$ of the singularly perturbed Cauchy problem (P_ϵ), when $\varepsilon \to 0$ within the boundary layer, then the function $Z(x,\tau)$ has to be reduced for the discrepancy elimination of the solutions $u(\varepsilon,x,0)$ and $u(0,x,0)$.

We can substitute expansion (3) into (1) formally and identify the coefficients of the same powers of ε, which contain the same variables.

Then we can get the following equations:

$$P_0 V_k = F_k(x,t), \quad x \in \mathbf{R}^d, \quad t > 0, \tag{4}$$

where

$$\begin{gathered} F_0 = f(x,t), \quad F_k = -P_1 V_{k-1}, \quad k = 1, \dots, N, \\ A_0 \partial_\tau Z_k = F_k(x,\tau), \quad k = 0, 1, \dots, N, \\ A_1(L_0 Z_N + L_1 Z_{N-1} + \partial_\tau Z_N) = 0, \quad x \in \mathbf{R}^d, \quad \tau > 0, \\ F_0 = 0, \quad F_1 = -L_0 Z_0 - A_1 \partial_\tau Z_0, \\ F_k = -L_0 Z_{k-1} - L_1 Z_{k-2} - A_1 \partial_\tau Z_{k-1}, \quad k = 2, \dots, N, \end{gathered} \tag{5}$$

$$\begin{gathered} (P_0 + \varepsilon P_1) R_N = F(x,t,\varepsilon), \quad x \in \mathbf{R}^d, \quad t > 0, \\ F = -\varepsilon^{N+1}(P_1 V_N + L_1 Z_N) - \varepsilon^N A_0(L_0 Z_N + L_1 Z_{N-1}). \end{gathered} \tag{6}$$

We can substitute (3) into initial condition (2)

$$R_N(\varepsilon, x, 0) = 0, \quad x \in \mathbf{R}^d, \tag{7}$$

$$V_0(x,0) + Z_0(x,0) = U_0(x), \quad x \in \mathbf{R}^d, \tag{8}$$

$$V_k(x,0) + Z_k(x,0) = 0, \, x \in \mathbf{R}^d, \quad k = 1, \dots, N. \tag{9}$$

We can use the following notation for convenience

$$Z_k = \begin{pmatrix} X_k \\ Y_k \end{pmatrix}, \quad V_k = \begin{pmatrix} v_k \\ w_k \end{pmatrix}, \quad F_k = \begin{pmatrix} f_k \\ d_k \end{pmatrix}, \quad F_k = \begin{pmatrix} F_{k1} \\ F_{k2} \end{pmatrix}, \tag{10}$$

where $X_k, v_k, f_k, F_{k1} \in M^{m\times 1}(\mathbf{R})$, $Y_k, w_k, g_k, F_{k2} \in M^{(n-m)\times 1}(\mathbf{R})$.

We can use (5), (8), and (9) for X_k and Y_k so that

$$\partial_\tau X_k = F_{k1}, \quad X_k \to 0, \quad \tau \to +\infty, \tag{11}$$

and

$$\begin{gathered} \partial_\tau Y_k + L_{03} Y_k = F_{k2}(x,\tau), \quad x \in \mathbf{R}^d, \quad \tau > 0, \\ Y_0(x,0) = u_1(x) - w_0(x,0), \quad x \in \mathbf{R}^d, \\ Y_k(x,0) = -w_k(x,0), \quad k = 1, \dots, N, \quad x \in \mathbf{R}^d, \end{gathered} \tag{12}$$

where

$$\begin{gathered} F_{01} = 0, \quad F_{11} = -L_{01} X_0 - L_{02} Y_0, \\ F_{k1} = -L_{01} X_{k-1} - L_{02} Y_{k-1} - L_{11} X_{k-2} - L_{12} Y_{k-2}, \quad k = 2, \dots, N, \\ F_{02} = -L_{02}^* X_0, \quad F_{k2} = -L_{02}^* X_k - L_{13} Y_{k-1} - L_{12}^* X_{k-1}, \quad k = 1, \dots, N \\ L_{ij}^*(\xi) = B_{ij}^*(\xi) + D_{ij}^*, \quad i = 0, 1, \quad j = 1, 2, 3. \end{gathered}$$

Similarly, we can obtain the problems for v_k and w_k from (4) and (8), (9).

$$\begin{cases} \partial_t v_k + L_{01} v_k + L_{02} w_k = f_k(x,t), \quad k = 0,1,\dots,N, \\ L_{02}^* v_k + L_{03} w_k = g_k(x,t), \quad x \in \mathbf{R}^d, \quad k = 0,1,\dots,N, \quad t > 0, \\ v_0(x,0) = u_0(x) - X_0(x,0), \\ v_k(x,0) = -X_k(x,0), \quad k = 1,\dots,N, \quad x \in \mathbf{R}^d. \end{cases} \tag{13}$$

Thus, we have the problems for determining the functions X_k, Y_k, v_k, w_k and R_N.

4. Justifying asymptotic expansions of the singularly perturbed Cauchy problem (P_ϵ)

We investigate the validity of the expansion (3) in the following sections. We can consider the problem (13) in the next form

$$\begin{cases} \partial_t v + L_{01} v + L_{02} w = f(x,t), \\ L_{02}^* v + L_{03} w = g(x,t), \quad x \in \mathbf{R}^d, \quad t > 0, \\ v(x,0) = h(x), \quad x \in \mathbf{R}^d, \end{cases} \tag{14}$$

$$L_{0j} = B_{0j}(\partial_x) + D_{0j} = \sum_{p=1}^{d} B_p^{0j} \partial_{x_p} - \sum_{p,q=1}^{d} C_{pq}^{0j} \partial_{x_p} \partial_{x_q} + D_{0j}, \quad j = 1,2,3.$$

We use the following problem for the solvability and regularity justifications of the problem (14)

$$\begin{cases} \partial_t \hat{v}(\xi) + (D_{01} + i|\xi|\hat{B}_{01}(\xi))\hat{v}(\xi) + (D_{02} + i|\xi|\hat{B}_{02}(\xi))\hat{w}(\xi) = \hat{f}(\xi,t), \\ (D_{02}^* + i|\xi|\hat{B}_{02}^*(\xi))\hat{v}(\xi) + (D_{03} + i|\xi|\hat{B}_{03}(\xi))\hat{w}(\xi) = \hat{g}(\xi,t), \\ \hat{v}(\xi,0) = \hat{h}(\xi), \end{cases}$$

$$\hat{B}_{ij}(\xi) = \sum_{p=1}^{d} B_p^{0j}(\xi_p/|\xi|) - i|\xi| \sum_{p,q=1}^{d} C_{pq}^{0j}(\xi_p \xi_q/|\xi|^2), \tag{15}$$

where $i = 0,1$, $j = 1,2,3$, $\xi \in \mathbf{R}^d$.

We prove the following lemmas.

Lemma 1. *The matrix $D_{03} + i|\xi|\hat{B}_{03}(\xi)$ is invertible for $\xi \in \mathbf{R}^d$ under the assumptions (**H1**), (**H2**) and the function $\xi \to (D_{03} + i|\xi|\hat{B}_{03}(\xi))^{-1}$ is bounded on $\mathbf{R}^d$.*

Proof. We can use the method of the simultaneous reduction of two matrices to the diagonal form for proving this lemma and we assume that

$D_{03}^* = D_{03}$ and $D_{03} = (d_{st}^0)_{s,t=\overline{n-m,n}}$, $d_{st}^0 \geqslant 0$ $(s,t=\overline{n-m,n})$. There is an orthogonal matrix $T_1 \in M^{n-m}(\mathbf{R})$, $T_1^* T_1 = I_{n-m}$, which

$$T_1^* D_{03} T_1 = \Lambda_0^2 = \mathrm{diag}(\lambda_1, \dots, \lambda_{n-m}),$$

where $\lambda_k > 0$, $k = 1, \dots, n-m$ are the eigenvalues of matrix D_{03}.

We can use the transformation of the matrix $\widehat{B}_{03}(\xi)$ in the form:

$$C(\xi) = \Lambda_0^{-1} T_1^* \widehat{B}_{03}(\xi) T_1 \Lambda_0^{-1}.$$

As the matrix $C(\xi)$ is a real symmetric, then there exists an orthogonal matrix $T_2(\xi) \in M(\mathbf{R}^{n-m})$, such that

$$T_2^* C(\xi) T_2 = \Lambda(\xi) = \mathrm{diag}(\mu_1(\xi), \dots, \mu_{n-m}(\xi)),$$

where $\mu_1(\xi), \dots, \mu_{n-m}(\xi)$ are real eigenvalues of matrix $C(\xi)$. Thus, we have the transformations of this type:

$$T^*(\xi) D_{03} T(\xi) = I_{n-m}, \quad T^*(\xi) \widehat{B}_{03}(\xi) T(\xi) = \Lambda(\xi), \tag{16}$$

where $T(\xi) = T_1 \Lambda_0^{-1} T_2(\xi)$. We can use (16) so that

$$D_{03} + i|\xi| \widehat{B}_{03}(\xi) = T^{*^{-1}}(\xi)(I_{n-m} + i|\xi|\Lambda(\xi)) T^{-1}(\xi).$$

It means that the matrix $D_{03} + i|\xi|\widehat{B}_{03}(\xi)$ is invertible and we have

$$(D_{03} + i|\xi|\widehat{B}_{03}(\xi))^{-1} = T(\xi)\Lambda_1(\xi)(I_{n-m} - i|\xi|\Lambda(\xi)) T^*(\xi), \tag{17}$$

where

$$\Lambda_1(\xi) = \mathrm{diag}((1 + |\xi|^2 \mu_1^2)^{-1}, \dots, (1 + |\xi|^2 \mu_{n-m}^2)^{-1}).$$

The orthogonality of the matrix $T_2(\xi)$ implies the boundedness of the function $\xi \to T(\xi)$ on $\mathbf{R}^d$.

The boundedness of the matrix $(D_{03} + i|\xi|\widehat{B}_{03}(\xi))^{-1}$ follows from (17). Lemma 1 is proved. $\square$

We can obtain the solution of the problem (15) from Lemma 1

$$\begin{cases} \dfrac{d}{dt}\hat{v}(\xi,t) + K(\xi)\hat{v}(\xi,t) = H(\xi,t), \\ \hat{v}(\xi,0) = \hat{h}(\xi), \end{cases} \tag{18}$$

where

$$\hat{w}(\xi,t) = (D_{03} + i|\xi|\widehat{B}_{03}(\xi))^{-1}(\hat{g}(\xi,t) - (D_{02}^* + i|\xi|\widehat{B}_{02}^*(\xi))\hat{v}(\xi,t)), \tag{19}$$

$$\begin{aligned} K(\xi) = D_{01} + i|\xi|\widehat{B}_{01}(\xi) - (D_{02} + i|\xi|\widehat{B}_{02}(\xi))(D_{03} + \\ + i|\xi|\widehat{B}_{03}(\xi))^{-1}(D_{02}^* + i|\xi|\widehat{B}_{02}^*(\xi)), \end{aligned} \tag{20}$$

$$H(\xi,t)=\hat{f}(\xi,t)-(D_{02}+i|\xi|\hat{B}_{02}(\xi))(D_{03}+i|\xi|\hat{B}_{03}(\xi))^{-1}\hat{g}(\xi,t).$$

Lemma 2. *The matrix $K(\xi)$ can be represented in the form*

$$K(\xi)=K_0(\xi)+i|\xi|K_1(\xi)+|\xi|^2K_2(\xi),\quad \xi\in\mathbf{R}^d, \tag{21}$$

*under the assumptions (**H1**), (**H2**), where the functions $\xi\to K_j(\xi)$, $j=0,1,2$ are bounded on $\mathbf{R}^d$ and K_1,K_2 are real symmetric for $K_2\geqslant 0$.*

Proof. Let us substitute (17) into (20). We can obtain the representation (21), where

$$K_0(\xi)=G_{01}-G_{02}T^*\Lambda_1T^*G_{02}^*-|\xi|^2(G_{02}T\Lambda_1\Lambda T^*b_{02}^*+b_{02}T\Lambda_1\Lambda T^*G_{02}^*),$$

$$\begin{aligned}K_1(\xi)=b_{01}+G_{02}T\Lambda_1\Lambda T^*G_{02}^*-G_{02}T\Lambda_1T^*b_{02}^*-&\\ -b_{02}T\Lambda_1T^*G_{02}^*-|\xi|^2b_{02}T\Lambda_1\Lambda T^*b_{02}^*,&\end{aligned}$$

$$K_2(\xi)=b_{02}T\Lambda_1T^*b_{02}^*.$$

Accordingly, $K_j(\xi)$, $j=0,1,2$ are bounded on $\mathbf{R}^d$ and $K_1^*=K_1$, $K_2^*=K_2$. It remains to prove that $K_2\geqslant 0$. Let us denote the eigenvalues of the real symmetric matrix A as $\lambda_j(A)$, $j=1,\dots,m$, where $\lambda_1\leqslant\lambda_2\leqslant\dots\leqslant\lambda_m$.

We can use Ostrowski's theorem so that

$$\lambda_j(K_2(\xi))=\lambda_j(b_{02}T\Lambda_1T^*b_{02}^*)=\theta_j\lambda_j(\Lambda_1)\geqslant 0,$$

where $0\leqslant\lambda_1(b_{02}TT^*b_{02}^*)\leqslant\theta_j\leqslant\lambda_m(b_{02}TT^*b_{02}^*)$. It means that $K_2\geqslant 0$. Therefore, Lemma 2 is proved. □

We can prove the following proposition.

Proposition 1. Let the assumptions (**H1**), (**H2**) be fulfilled and $l\in\mathbb{N}^*$. If the conditions $h\in H_m^{s+2l+1}$, $F=\mathrm{col}(f,g)\in W^{l,1}(0,T;H_n^{s+2})$ are true, then there exists a unique strong solution $V=\mathrm{col}(v,w)\in W^{l,\infty}(0,T;H_n^s)$ of the problem (14) and

$$\|V\|_{W^{l,\infty}(0,T;H_n^s)}\leqslant C(T)\left(\|h\|_{s+2l+1,m}+\|F\|_{W^{l,1}(0,T;H_n^{s+2})}\right). \tag{22}$$

Proof. Consider the Cauchy problem

$$\begin{cases}\frac{d}{dt}\hat{v}(t)+K(\xi)\hat{v}(t)=0,\\ \hat{v}(0)=\hat{h},\quad 0<t<T,\end{cases} \tag{23}$$

in the Hilbert space $H=\{f=(f_1,\dots,f_m);\,(1+|\xi|^2)^{\frac{s}{2}}f_k(\xi)\in L^2(\mathbf{R}^d),\ k=1,\dots,m\}$, equipped with the scalar product $(f,g)_H=\int_{\mathbf{R}^d}(1+$

$|\xi|^2)^s(f,\bar{g})_{\mathbf{R}^m}\,d\xi$. We can use the representation (21) and demonstrate that the operator $-K(\xi): H \to H$ satisfies the conditions

$$\mathrm{Re}(-Kf,f)_H \leqslant \omega(f,f)_H,\ \mathrm{Re}(-\bar{K}^*f,f)_H \leqslant \omega(f,f)_H, \quad f\in H,$$

where $\omega = sup_{\xi\in\mathbf{R}^d}\|K_0(\xi)\|_{\mathbf{R}^m\to\mathbf{R}^m} + \delta$ with a positive parameter $\delta>0$. This means that the operator $-(K+\omega I)$ is maximal dissipative on H.

The Cauchy problem (23) generates a C_0 semigroup of operators $\{\hat{T}(t),\ t\geqslant 0\}$ on H [21]. Thus, we have the next estimation $\|\hat{v}(\cdot,t)\|_H \leqslant e^{\omega t}\|h\|_H$ for any $h\in H$, i.e. $\|\hat{T}(t)\|\leqslant e^{\omega t}$, where

$$\frac{d}{dt}\|\hat{v}(\cdot,t)\|_H^2 \leqslant -(K_0\hat{v}(\cdot,t),\hat{v}(\cdot,t))_H - (\hat{v}(\cdot,t),K_0\hat{v}(\cdot,t))_H \leqslant 2\omega\|\hat{v}(\cdot,t)\|_H^2.$$

Using Parseval's equality, we can get that the Cauchy problem ($F[\check{K}v] = K(\xi)\hat{v}$)

$$\begin{cases}\dfrac{d}{dt}v(t)+\check{K}v(t)=0,\\ v(0)=v_0,\quad 0<t<T,\end{cases}\tag{24}$$

where the operators $\{T(t), t\geqslant 0\}$ on H_m^s generates the semigroup C_0, where $v(\cdot,t)=T(t)v_0$, $\|T(t)\|\leqslant e^{\omega t}$. Thus, we can solve the Cauchy problem

$$\begin{cases}\dfrac{d}{dt}z(t)+(\check{K}+\omega I)z(t)=f(t)e^{\omega t},\\ z(0)=y_0,\quad 0<t<T,\end{cases}\tag{25}$$

where the semigroup C_0 has the representation in the form $T_0(t)=T(t)e^{-\omega t}$.

Hence, there exists a unique mild solution of this problem $z\in C([0,T];H_m^s)$ for every $y_0\in H_m^s$, $f\in L^1(0,T;H_m^s)$ [21], and

$$z(t)=T_0(t)y_0+\int_0^t T_0(t-s)f(s)e^{\omega s}ds,$$

$$\|z\|_{C([0,T];H_m^s)} \leqslant \|y_0\|_{s,m}+\|f\|_{L^1(0,T;H_m^s)}e^{\omega T}.$$

Moreover, if the next $y_0\in H_m^{s+2l}$, $f\in W^{l,1}(0,T;H_m^s)$ and $l\in\mathbb{N}^*$ are true, then z is a strong solution of the problem (25), $z\in W^{l,\infty}(0,T;H_m^s)$ and

$$\|z\|_{W^{l,\infty}(0,T;H_m^s)} \leqslant C(T)(\|y_0\|_{s+2l,m}+\|f\|_{W^{l,1}(0,T;H_m^s)}).$$

We can note that the solution y of the Cauchy problem

$$\begin{cases}\frac{d}{dt}y(t)+\check{K}y(t)=f,\\ y(0)=y_0,\quad 0<t<T,\end{cases}\tag{26}$$

and the solution z of the problem (25) are connected with the equality $y(t) = e^{-\omega t} z(t)$.

Consequently, we have the same for y_0, f and $l \in N^*$ so that

$$\|y\|_{W^{l,\infty}(0,T;H^s_m)} \leqslant C(T)(\|y_0\|_{s+2l,m} + \|f\|_{W^{l,1}(0,T;H^s_m)}).$$

Using (18), the last estimation and the boundedness of the matrix $(G_{03} + i|\xi|b(\xi))^{-1}$, we can obtain the next estimation

$$\|v\|_{W^{l,\infty}(0,T;H^s_m)} \leqslant \\ \leqslant C(T)(\|h\|_{s+2l,m} + \|f\|_{W^{l,1}(0,T;H^s_m)} + \|g\|_{W^{l,1}(0,T;H^{s+1}_{n-m})}). \quad (27)$$

We can get the estimation from (19) and (27) in the form:

$$\|w\|_{W^{l,\infty}(0,T;H^s_m)} \leqslant \\ \leqslant C(T) \left(\|h\|_{s+2l+1,m} + \|f\|_{W^{l,1}(0,T;H^{s+1}_m)} + \|g\|_{W^{l,1}(0,T;H^{s+2}_{n-m})}\right). \quad (28)$$

Thus, the estimations (27) and (28) imply the estimation (23). Proposition 1 is proved. □

Let us consider the next Cauchy problem

$$\begin{cases} \partial_\tau Y + L_{03} Y = F(x,\tau),\, x \in \mathbf{R}^d, \quad \tau > 0, \\ Y(x,0) = y_0(x), \quad x \in \mathbf{R}^d. \end{cases} \quad (29)$$

Proposition 2. Let the assumptions (**H1**), (**H2**) be fulfilled and $l \in \mathbb{N}^*$. If the conditions $y_0 \in H^{s+l}_{n-m}$, $F \in W^{l,1}_{\text{loc}}(0,\infty; H^s_{n-m})$ are true, then there exists a unique strong solution $Y \in W^{l,\infty}_{\text{loc}}(0,\infty; H^s_{n-m})$ of the problem (29) and the inequality is satisfied for this solution

$$\|\partial^l_\tau Y(\cdot,\tau)\|_{s,n-m} \leqslant Ce^{-q_0\tau}(\|y_0\|_{s+l,n-m} + \\ + \sum_{\nu=0}^{l-1} \|\partial^\nu_\tau F(\cdot,0)\|_{s+l-\nu-1,n-m} + \int_0^\tau e^{q_0\theta}\|\partial^l_\tau F(\cdot,\theta)\|_{s,n-m}\, d\theta). \quad (30)$$

Proof. The operator $-L_{03}(\partial_x)$ is a dissipative under the assumptions (**H1**), (**H2**) and it generates the C_0 semigroup of the contractions $S(\tau)$ on H^s_{n-m}. Thus, there exists a unique mild solution $Y \in C([0,\infty); H^s_{n-m})$ of the Cauchy problem (29). Hence, we can obtain the estimation $\|S(\tau)\| \leqslant e^{-q_0\tau}$, $\tau \geqslant 0$, which with the next equality

$$Y(\cdot,\tau) = S(\tau)y_0 + \int_0^\tau S(\theta)F(\cdot,\tau-\theta)\, d\theta$$

gives the estimation (30) in the case $l = 0$. We can obtain the estimation (30) by differentiating to τ the equation (29) in the case $l \geqslant 1$. Proposition 2 is proved. □

Using these propositions, we can determine the functions V_k and Z_k. Hence, it follows from (11) for $k = 0$ that $X_0 = 0$. We can find the main regular term $V_0 = \mathrm{col}(v_0, w_0)$ of the expansion (3) from (13) and Proposition 1. Instantly, we have the following:

$$w_0(x,0) = F^{-1}[(G_{03} + i|\xi|b_{03}(\xi))^{-1}(\hat{g}(\xi,0) - (G^*_{02} + i|\xi|b^*_{02}(\xi))\hat{u}_0(\xi))].$$

Lemma 1 and the Parseval equality permit us to obtain the next estimation

$$\|w_0(\cdot,0)\|_{s,n-m} \leqslant C(\|g(\cdot,0)\|_{s,n-m} + \|u_0\|_{s+1,m}) \leqslant \\ \leqslant C(\|U_0\|_{s+1,n} + \|F(\cdot,0)\|_{s,n}). \quad (31)$$

Proposition 2 permits us to define the function Y_0 as a solution of Cauchy problem (12). Moreover, we can obtain the next inequality from (30) and (31)

$$\|\partial^l_\tau Y_0(\cdot,\tau)\|_{s,n-m} \leqslant Ce^{-q_0\tau}(\|U_0\|_{s+l+1,n} + \|F(\cdot,0)\|_{s+l,n}). \quad (32)$$

Thus, we can find the main singular term $Z_0 = \mathrm{col}(0, Y_0)$ of the expansion (3).

Let us obtain the next terms of this expansion. Let us suppose that the terms $V_0, \ldots, V_{k-1}$ and $Z_0, \ldots, Z_{k-1}$ are already found. We can obtain the terms V_k and Z_k and show that the next estimations

$$\|V_k\|_{W^{l,\infty}(0,T;H^s_n)} \leqslant C(T)(\|U_0\|_{s+2l+3k+1,n} + \\ + \|F(\cdot,0)\|_{s+2l+3k-2,n} + \|F\|_{W^{l,1}(0,T;H^{s+3k+2}_n)}), \quad (33)$$

and

$$\|\partial^l_\tau Z_k(\cdot,\tau)\|_{s,n} \leqslant Ce^{-q_0\tau}(1+\tau^k)\left(\|U_0\|_{s+l+k+1,n} + \|F(\cdot,0)\|_{s+l+k,n}\right) \quad (34)$$

are true, if we suppose that such estimations are true for previous terms. We can note that the estimations (33), (34) for V_0 and Z_0 follow from (22) and (32).

At first, if we solve the problem (11), we can get

$$X_k(\cdot,\tau) = -\int_\tau^\infty F_{k1}(\cdot,\theta)\,d\theta,$$

where the integral exists due to the estimation (34) for Z_{k-1}. Using (34) for Z_{k-1} and for Z_{k-2}, we obtain the next estimation:

$$\|\partial^l_\tau X_k(\cdot,\tau)\|_{s,m} = \|\partial^{l-1}_\tau F_{k1}(\cdot,\tau)\|_{s,m} \leqslant \\ \leqslant C(\|\partial^{l-1}_\tau Z_{k-1}(\cdot,\tau)\|_{s+1,n} + \|\partial^{l-1}_\tau Z_{k-2}(\cdot,\tau)\|_{s+1,n}) \leqslant \\ \leqslant Ce^{-q_0\tau}(1+\tau^{k-1})(\|U_0\|_{s+l+k,n} + \|F(\cdot,0)\|_{s+l+k-1,n}), \quad (35)$$

for $l \geqslant 1$. Similarly, we can get the estimation (35) in the case $l = 0$.

Using Proposition 1 and $v_k(\cdot,0) = -X_k(\cdot,0)$, we can solve the problem (13) and find the functions V_k.

Using the next estimation

$$\|V_k\|_{W^{l,\infty}(0,T;H^s_n)} \leqslant C(T)(\|X_k(\cdot,0)\|_{s+2l+1,m} + \|V_{k-1}\|_{W^{l,\infty}(0,T;H^{s+3}_n)}),$$

and also (22), (33) for V_{k-1} and (35) for X_k, we can find the estimation (33) for V_k.

Instantly, we can obtain the next equality

$$w_k(x,0) = F^{-1}[(D_{03} + i|\xi|\hat{B}_{03}(\xi))^{-1}(\hat{g}_k(\xi,0) - (D^*_{02} + i|\xi|\hat{B}^*_{02}(\xi))\hat{X}_k(\xi,0))]$$

and establish the estimation

$$\begin{aligned} \|w_k(\cdot,0)\|_{s,n-m} &\leqslant C(\|g_k(\cdot,0)\|_{s,n-m} + \|X_k(\cdot,0)\|_{s+1,m}) \leqslant \\ &\leqslant C(\|X_{k-1}(\cdot,0)\|_{s+1,m} + \|X_k(\cdot,0)\|_{s+1,m} + \|w_{k-1}(\cdot,0)\|_{s+1,n-m}) \leqslant \\ &\leqslant C(\|U_0\|_{s+k+1,n} + \|F(\cdot,0)\|_{s+k,n}). \end{aligned} \quad (36)$$

Using (34) for Z_{k-1} and (35) for X_k, we can obtain the next unequality

$$\begin{aligned} \|\partial^l_\tau F_{k2}(\cdot,\tau)\|_{s,n-m} &\leqslant C(\|\partial^l_\tau X_k(\cdot,\tau)\|_{s+1,m} + \|\partial^l_\tau Z_{k-1}(\cdot,\tau)\|_{s+1,n}) \leqslant \\ &\leqslant Ce^{-q_0\tau}(1+\tau^{k-1})(\|U_0\|_{s+l+k+1,n} + \|F(\cdot,0)\|_{s+l+k,n}). \end{aligned} \quad (37)$$

We can find the next estimation from (30), (36) and (37)

$$\begin{aligned} \|\partial^l_\tau Y_k(\cdot,\tau)\|_{s,n-m} &\leqslant Ce^{-q_0\tau}(\|w_k(\cdot,0)\|_{s+l,n-m} + \\ &+ \sum_{\nu=0}^{l-1} \|\partial^\nu_\tau F_{k2}(\cdot,0)\|_{s+l-\nu-1,n-m} + \int_0^\tau e^{q_0\theta}\|\partial^l_\tau F_{k2}(\cdot,\theta)\|_{s,n-m}\,d\theta) \leqslant \\ &\leqslant Ce^{-q_0\tau}(1+\tau^k)(\|U_0\|_{s+l+k+1,n} + \|F(\cdot,0)\|_{s+l+k,n}). \end{aligned} \quad (38)$$

The estimations (35) and (38) imply the estimation (34) for Z_k.

We can prove the main result of our work.

Theorem 1. *Let us suppose that B and G satisfy conditions (**H1**), (**H2**) and $0 \leqslant l < N+1$. If the conditions $U_0 \in H^{s+2l+3(N+1)}_n$, $F \in W^{l+1,1}(0,T;H^{s+2l+3(N+1)}_n)$ are true, then there exists a unique strong solution $U \in W^{l,\infty}(0,T;H^s_n)$ of the problem (P_ε). The expansion (3) is true for this solution, where V_k and Z_k are determined by problems (13), (11), (12) respectively and they satisfy the estimations (33), (34). The estimation*

$$\|R_{N1}\|^2_{W^{l,\infty}(0,T;H^s_m)} + \varepsilon^{1/2}\|R_{N2}\|^2_{W^{l,\infty}(0,T;H^s_{n-m})} \leqslant C(T)\varepsilon^{N+1-l} \quad (39)$$

is true with $C(T)$ depending on T, $\|U_0\|_{s+2l+3(N+1),n}$, $\|F\|_{W^{l+1,1}(0,T;H^{s+2l+3(N+1)}_n)}$ and q_0 for the remainder term $R_N = \mathrm{col}(R_{N1}, R_{N2})$. In particular, if we assume $N = 0$, then there is the next estimation

$$\|U - V_0 - Z_0\|_{C([0,T];H^s_n)} \leqslant C(T)\varepsilon^{1/4}.$$

Proof. Using the properties of the C_0 semigroup of operators, we can obtain the solvability of the problem (P_ε). Indeed, the operator $-(B(\partial_x)+D)$ is closed and dissipative on H_n^s. This operator generates the C_0 semigroup of contractions on H_n^s, which solves the problem (P_ε). Moreover, the conditions $U_0 \in H_n^{s+l}$, $F \in W^{l,1}(0,T;H_n^s)$, $\partial_t^\nu F(\cdot,0) \in H_n^{s+l-\nu-1}$, $\nu = 0,\dots,l-1$, $l \geqslant 1$ imply the regularity of the solution $U \in W^{l,\infty}(0,T;H_n^s)$. Using the method from [21], we can prove the estimation (39). Furthermore, all constants depend on the norms, which are indicated in the Theorem 1, and they are represented by $C(T)$. Let us denote the next relations $R_l = \partial_t^l R_N$, $R_{li} = \partial_t^l R_{Ni}$, $i = 1,2$. We can find that $(BR_l, R_l)_{s,n}$ is a pure imaginary value from the condition (**H1**). Consequently, we can get the next equation

$$\frac{d}{dt}(AR_l(\cdot,t),R_l(\cdot,t))_{s,n} + 2(GR_l(\cdot,t),R_l(\cdot,t))_{s,n} = 2Re(\partial_t^l F(\cdot,t),R_l(\cdot,t))_{s,n}.$$

Using the assumption $(H2)$, we can get the next inequality

$$\frac{d}{dt}(AR_l(\cdot,t),R_l(\cdot,t))_{s,n} + 2q_0(R_{l2}(\cdot,t),R_{l2}(\cdot,t))_{s,n-m} \leqslant \\ \leqslant 2|(\partial_t^l F(\cdot,t),R_l(\cdot,t))_{s,n}|. \quad (40)$$

The estimations (33) and (34) yield the next estimation

$$|(\partial_t^l F(\cdot,t),R_l(\cdot,t))_{s,n}| \leqslant \varepsilon^{N+1}|(P_1(\partial_t^l V_N(\cdot,t)) + \varepsilon^{-l}L_1(\partial_\tau^l Z_N(\cdot,\tau)), \quad (41)$$

$$R_l(\cdot,t))_{s,n}| + \varepsilon^{N-l}|(L_0(\partial_\tau^l Z_N(\cdot,\tau)) + L_1(\partial_\tau^l Z_{N-1}(\cdot,\tau)),$$

$$A_0R_l(\cdot,t))_{s,n}| \leqslant \\ \leqslant C(T)(\varepsilon^{N-l}\kappa(t)\|R_{l1}(\cdot,t)\|_{s,m} + (\varepsilon^{N+1} + \kappa(t)\varepsilon^{N+1-l})\|R_l(\cdot,t))\|_{s,n}),$$

where $0 \leqslant t \leqslant T, \tau = t/\varepsilon$ and $\kappa(t) = e^{-q_0 t/\varepsilon}(1+(t/\varepsilon)^N)$. Integrating (40) by t and using (41), we can get the next inequality

$$\|R_{l1}(\cdot,t))\|^2_{s,m} + \varepsilon\|R_{l2}(\cdot,t))\|^2_{s,n-m} + 2q_0\int_0^t \|R_{l2}(\cdot,\theta)\|^2_{s,n-m}\,d\theta \leqslant \\ \leqslant \|R_{l1}(\cdot,0)\|^2_{s,m} + \varepsilon\|R_{l2}(\cdot,0))\|^2_{s,n-m} + C(T)(\varepsilon^{N-l}\int_0^t \kappa(\theta)\|R_{l1}(\cdot,\theta)\|_{s,m}\,d\theta + \\ + \int_0^t (\varepsilon^{N+1} + \kappa(\theta)\varepsilon^{N-l+1})\|R_l(\cdot,\theta)\|_{s,n}\,d\theta), \quad 0 \leqslant t \leqslant T, \quad (42)$$

We can note that

$$R_l(\cdot,0) = \sum_{\nu=0}^{l-1}(-A^{-1}(B(\partial_x)+D))^{l-\nu-1}A^{-1}\partial_t^\nu F(\cdot,0), \quad l \geqslant 1,$$

and according to (7), $R_0(\cdot,0)=0$.

Therefore, using the equality $A^{-1}A_0 = A_0$ and (34), (35), we can find the next estimation

$$\begin{aligned}
&\|A^{-1}\partial_t^\nu F(\cdot,0)\|_{s,n} \leqslant \\
&\quad\leqslant \varepsilon^{N+1}\|(A^{-1}P_1\partial_t^\nu V_N)(\cdot,0)\|_{s,n} + \varepsilon^{N+1-\nu}\|(A^{-1}L_1\partial_\tau^\nu Z_N)(\cdot,0)\|_{s,n} + \\
&\qquad + \varepsilon^{N-\nu}\|A_0(L_0\partial_\tau^\nu Z_N + L_1\partial_\tau^\nu Z_{N-1})(\cdot,0)\|_{s,n} \leqslant \\
&\qquad\qquad \leqslant C(T)(\varepsilon^N + \varepsilon^{N-\nu}) \leqslant C(T)\varepsilon^{N-\nu},
\end{aligned}$$

where $0<\varepsilon<1$, $0\leqslant \nu \leqslant N$.

Thus, we can obtain the next inequalities

$$\begin{aligned}
\|R_l(\cdot,0)\|_{s,n} &\leqslant \sum_{\nu=0}^{l-1} \|A^{-1}(B(\partial_x)+D))^{l-\nu-1}A^{-1}\partial_t^\nu F(\cdot,0)\|_{s,n} \leqslant \\
&\leqslant C(T)\sum_{\nu=0}^{l-1} \varepsilon^{-(l-\nu-1)}\cdot\varepsilon^{N-\nu} \leqslant C(T)\varepsilon^{N-l+1}. \quad (43)
\end{aligned}$$

If the conditions $l<N+1$, $0\leqslant t\leqslant T$, $\varepsilon \ll 0$ are true, we can obtain the estimations

$$\begin{aligned}
\int_0^t \kappa(\theta)\|R_{l1}(\cdot,\theta)\|_{s,m}\,d\theta &\leqslant \int_0^t \kappa(\theta)\,d\theta + \int_0^t \kappa(\theta)\|R_{l1}(\cdot,\theta)\|_{s,m}^2\,d\theta \leqslant \\
&\leqslant C(T)\varepsilon + \int_0^t \kappa(\theta)\|R_{l1}(\cdot,\theta)\|_{s,m}^2\,d\theta, \quad (44)
\end{aligned}$$

and

$$\begin{aligned}
&C(T)\int_0^t (\varepsilon^{N+1}+\kappa(\theta)\varepsilon^{N-l+1})\|R_l(\cdot,\theta)\|_{s,n}\,d\theta \leqslant \\
&\qquad \leqslant C(T)\varepsilon^{N-l+1} + q_0\int_0^t \|R_{l2}(\cdot,\theta)\|_{s,n-m}^2\,d\theta + \\
&\qquad\quad + C(T)\int_0^t (\varepsilon^{N+1}+\kappa(\theta)\varepsilon^{N-l+1})\|R_{l1}(\cdot,\theta)\|_{s,m}^2\,d\theta. \quad (45)
\end{aligned}$$

Using the next inequality

$$\begin{aligned}
&\|R_{l1}(\cdot,t))\|_{s,m}^2 + \varepsilon\|R_{l2}(\cdot,t))\|_{s,n-m}^2 + q_0\int_0^t \|R_{l2}(\cdot,\theta)\|_{s,n-m}^2\,d\theta \leqslant \\
&\quad \leqslant C(T)(\varepsilon^{N-l+1} + \int_0^t (\varepsilon^{N+1}+\kappa(\theta)\varepsilon^{N-l})\|R_{l1}(\cdot,\theta)\|_{s,m}^2\,d\theta),\ 0\leqslant t\leqslant T,
\end{aligned}$$

and the estimations (43), (44), (45), we can find the inequality (42).

Using Gronwall's lemma and the last inequality, we can get the estimations

$$\|R_{l1}(\cdot,t)\|^2_{s,m} \leqslant C(T)\varepsilon^{N-l+1},\ 0 \leqslant t \leqslant T, \tag{46}$$

and

$$\varepsilon\|R_{l2}(\cdot,t)\|^2_{s,n-m} + q_0 \int_0^t \|R_{l2}(\cdot,\theta)\|^2_{s,n-m}\, d\theta \leqslant C(T)\varepsilon^{N-l+1},\ 0 \leqslant t \leqslant T. \tag{47}$$

Using (43) and (47), we can obtain the estimation

$$\begin{aligned}
&\|R_{l2}(\cdot,t)\|^2_{s,n-m} \leqslant \\
&\quad \leqslant \|R_{l2}(\cdot,0)\|^2_{s,n-m} + 2\int_0^t \|R_{l2}(\cdot,\theta)\|_{s,n-m}\|R_{(l+1)2}(\cdot,\theta)\|_{s,n-m}\, d\theta \leqslant \\
&\qquad \leqslant C(T)\varepsilon^{2(N-l+1)} + 2\left(\int_0^t \|R_{l2}(\cdot,\theta)\|^2_{s,n-m}\, d\theta\right)^{1/2} \times \\
&\quad \times \left(\int_0^t \|R_{(l+1)2}(\cdot,\theta)\|^2_{s,n-m}\, d\theta\right)^{1/2} \leqslant C(T)\varepsilon^{N-l+1/2}, \quad 0 \leqslant t \leqslant T.
\end{aligned} \tag{48}$$

The estimates (46) and (48) imply the estimate (39). Therefore, Theorem 1 is proved. □

Thus, we justify asymptotic expansions of the singularly perturbed Cauchy problem (P_ϵ).

5. Conclusions

In this paper we investigate the Cauchy problem for the singularly perturbed Tikhonov-type symmetric system of Fokker–Planck equations. This system consists of non-homogeneous constant coefficients linear parabolic partial differential equations with a small parameter. For these singularly perturbed Cauchy problems a method for constructing asymptotic solutions is proposed. We use the asymptotic method for this Cauchy problem and construct expansions of solutions in the form of decomposition, which has regular and border-layer parts. The asymptotic solutions in the form of regular and boundary-layer parts are obtained and the question of asymptotic solutions behavior when $\varepsilon \to 0$ is investigated. The main result of our work is a justification of an asymptotic expansion for this Cauchy problem. We prove the justification theorem for the asymptotic solutions. Our method can be applied in a wide variety of cases for singularly perturbed Cauchy problems of Fokker–Planck equations. The Fokker–Planck equation is connected with the Chapman–Kolmogorov equation for the transition probability function of a Markov process.

Our results give the approach to investigate the fast-changing processes in liquids and gases, plasma, solid state theory, magnetic, hydrodynamics, radiophysics, telecommunication technology, chemistry, biology, finance and so on. An extension of Fokker–Planck equations with a small parameter to model non-Markovian processes is also possible.

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Асимптотическое решение сингулярно возмущённой задачи Коши для уравнения Фоккера–Планка

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Асимптотические методы — очень важная область прикладной математики. Существует множество современных направлений исследований, в которых используется малый параметр, например статистическая механика, теория химических реакций и др. Использование уравнения Фоккера–Планка с малым параметром очень востребовано, поскольку это уравнение является параболическим дифференциальным уравнением в частных производных, а решения этого уравнения дают функцию плотности вероятности.

В работе исследуется сингулярно возмущённая задача Коши для симметричной линейной системы параболических дифференциальных уравнений в частных производных с малым параметром. Мы предполагаем, что эта система является неоднородной системой тихоновского типа с постоянными коэффициентами. Цель исследования — рассмотреть эту задачу Коши, применить асимптотический метод и построить асимптотические разложения решений в виде двухкомпонентного ряда. Таким образом, это разложение имеет регулярную и погранслойную части. Основным результатом данной работы является обоснование асимптотического разложения для решений этой задачи Коши. Наш метод может быть применён для широкого круга сингулярно возмущённых задач Коши для уравнений Фоккера–Планка.

Ключевые слова: асимптотический анализ, сингулярно возмущённое дифференциальное уравнение, задача Коши, уравнение Фоккера–Планка