Discrete & Continuous Models
& Applied Computational Science

2019, 27 (2) 105–123
http://journals.rudn.ru/miph

Computational modeling and simulation

Research article

UDC 512, 519.6
DOI: 10.22363/2658-4670-2019-27-2-105-123

On algebraic integrals of a differential equation

Mikhail D. Malykh[1], **Leonid A. Sevastianov**[1], **Yu Ying**[1,2]

[1] *Department of Applied Probability and Informatics*
Peoples' Friendship University of Russia (RUDN University)
6, Miklukho-Maklaya St., Moscow, 117198, Russian Federation
[2] *Department of Algebra and Geometry*
Kaili University
3, Kaiyuan Road, Kaili, 556011, China

(received: October 24, 2019; accepted: November 14, 2019)

We consider the problem of integrating a given differential equation in algebraic functions, which arose together with the integral calculus, but still is not completely resolved in finite form. The difficulties that modern systems of computer algebra face in solving it are examined using Maple as an example. Its solution according to the method of Lagutinski's determinants and its implementation in the form of a Sagemath package are presented.

Necessary conditions for the existence of an integral of contracting derivation are given. A derivation D of the ring A will be called contracting, if such basis $B = \{m_1, m_2, \dots\}$ exists in which $Dm_i = c_i m_i + o(m_i)$. We prove that a contracting derivation of a polynomial ring R admits a general integral only if among the indices $c_1, c_2, \dots$ there are equal ones. This theorem is convenient for applying to the problem of finding an algebraic integral of Briot–Bouquet equation and differential equations with symbolic parameters. A number of necessary criteria for the existence of an integral are obtained, including those for differential equations of the Briot and Bouquet. New necessary conditions for the existence of a rational integral concerning a fixed singular point are given and realized in Sage.

Key words and phrases: Darboux polynomials; algebraic integrals of differential equations; finite solution; Sage; Sagemath; Maple

1. De Beaune problem

In the theory of differential equations, it is common from the very beginning to choose a class of functions in which solutions of differential equations are sought so wide that the initial problem has solutions for almost all initial data. In the case of symbolic integration, or finding the solution in finite form, on the contrary, this class is constricted to make it possible in a finite number of operations, first, to find out whether the given differential equation has a general solution in this class, and second, to write out this solution explicitly. The simplest class, which could be expected to possess the above two properties, is the set of algebraic functions.

The problem of integrating differential equations in algebraic functions arose as early as the 1630s, when Forimond de Beaune proposed to Descartes several "inverse tangent problems" [1, Pp. 510–518]. We formulate this purely algebraic problem as follows.

Problem 1 (de Beaune). *Clarify whether a given differential equation*

$$p(x,y)dx + q(x,y)dy = 0, \quad p, q \in k[x,y], \tag{1}$$

has an integral r in the field $k(x,y)$; in the case of a positive answer, write out this integral.

Here k is the field of constants, commonly represented by $\mathbb{Q}$, $\mathbb{C}$ or $\mathbb{Q}[a,b,\dots]$, where $a,b,\dots$ are the parameters that enter the differential equation. There is no reason to consider these cases separately, so we assume that k is an infinite field of characteristic zero.

The interest to the De Beaune problem sometimes faded away, sometimes arose again. At the turn of the XIX–XX centuries, it was due to successes in proving the nonexistence of algebraic integrals of dynamical systems; among the papers of this period worth particular attention are the Poincaré memoir [2, Pp. 35–95] and a series of articles by M.N. Lagutinski [3,4]; the biographical data were published by J.-M. Strelcyn [5,6].

Recently, the classical problem of finding an algebraic integral has again become relevant in connection with the development of algorithms for the symbolic solution of differential equations suitable for implementation in modern computer algebra systems [7,8]. First of all, it should be noted that popular computer algebra systems cannot efficiently recognize differential equations having algebraic integrals.

Example 1. To confirm this statement the following test was used. Let u, v — be arbitrary polynomials, then $w = u/v$ is an integral of the differential equation

$$\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right)dx + \left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right)dy = 0.$$

Taking randomly u and v, we get the differential equation

$$p(x,y)dx + q(x,y)dy = 0, \quad p, q \in \mathbb{Q}[x,y].$$

An attempt to apply standard methods of solving differential equations to this differential equation in the Maple computer algebra system reduces the differential equation to a quadrature of the form

$$\int r dx + s dy = C, \quad r, s \in \mathbb{Q}(x,y),$$

occupying many screens, moreover, Maple cannot take the written integrals.

It is worth noting that for the symbolic solution of differential equations in Maple the package DETools [9] is used. Within the second algorithm of DETools the search for integrating factors in the ring $\mathbb{C}[x, y]$ is executed. The equation generated in the test has several such integrating factors, namely, u and v, so that Maple would have to cope with the test. However, the following occurs:

— `symgen` returns two integrating factors, whose ratio yields the rational desired integral,
— `dsolve` ignores the second factor and write out a quadrature which it cannot calculate in elementary functions, although the full implementation of Ostrogradski algorithm would cope with this difficulty.

Thus, usually Maple cannot recognize an algebraic integral, however, the user can do it himself, looking at the result of applying the function `symgen`.

Insurmountable difficulties arise when p and q have common factors. The methods implemented by Maple, first of all, relieve the ordinary differential equation to be solved from common factors. The reduced equation may not have integrating factors in the ring $\mathbb{C}[x, y]$, and finding factors from $\mathbb{C}(x, y)$ leads to nonlinear equations for the coefficients and requires completely different computational costs for which the developers of `symgen` did not go. As a result, e.g., when

$$u = (x^2 + y)^5(x - y^6 + 1) + 1, \quad v = (13xy^8 + y^5 + 3xy + 2)(x^2 + y)^4,$$

`symgen` finds one factor from $\mathbb{Q}[x, y]$ and nothing else.

Despite the antiquity of the de Beaune problem, we do not have an algorithm to solve it in a finite number of operations. The de Beaune problem is equivalent to the problem of integrating a partial differential equation

$$p\frac{\partial r}{\partial y} - q\frac{\partial r}{\partial x} = 0$$

in the field $k(x, y)$; we will further briefly write it as $Dr = 0$. By the method of uncertain coefficients, we can substitute into the equation $Dr = 0$ the expression

$$r = \frac{a + \dots + by^n}{1 + \dots + cy^n}$$

and obtain a system of nonlinear algebraic equations for finding the coefficient $a, b, c, \dots$. The solvability of this system can be determined in a finite number of steps and in a purely algebraic way. Therefore, in a finite number of operations one can find out whether a given differential equation has rational integrals whose degree does not exceed a given number n.

The problem of finding the upper bound for the degree of the sought integral was noted by Descartes, and in some cases was resolved by Poincaré [2], pp. 35-95. The idea of the Poincaré method is as follows. If a differential equation admits a rational integral, then its integral curves form a linear sheaf of algebraic curves of some order n, this immediately follows from a comparison of the Cauchy theorem from the analytic theory of differential equations [10] and Bertini's theorem from the theory of algebraic curves [11]. Two arbitrary curves of the sheaf intersect at n^2 fixed points. On the other hand, according to the Cauchy theorem, these curves can intersect only at those points at which the polynomials p and q vanish simultaneously; in the analytical theory of differential equations, such singular points are called fixed points. If the orders of the curves $p(x, y) = 0$ and $q(x, y) = 0$ do not exceed m, then $n \leqslant m$. However, it is impossible to bring this idea to a rigorous statement: among the intersection points of the integral curves there may be multiple and infinitely distant ones, as well as at fixed singular points of the differential equation, the solutions may have various kinds of "degeneracies". That is why M.N. Lagutinski carefully notes that the "French scientist in the work just referred deduces a number of equalities and inequalities that in some cases achieve the goal of indicating the upper bound of the order n" [3, P. 181]. Taking into account that "the difficulties of this way for solving this problem have stopped even H. Poincaré" [3], it is not hard to understand why in all modern implementations of algorithms for finding integrals, the order of the integral is assumed to be given [12].

The de Beaune problem, in which a bound for the orders of considered integrals is given, will be referred to as a bounded problem.

Problem 2 (The bounded de Beaune problem). *Clarify whether a given differential equation*

$$p(x, y)dx + q(x, y)dy = 0, \quad p, q \in k[x, y], \tag{2}$$

admits an integral r in the field $k(x, y)$ whose order does not exceed a given number N, and in case of positive answer, write out this integral.

Practically the described solution of a system of nonlinear algebraic equations requires considerable computation resources even at $N = 3$. Therefore, the authors of algorithms for solving this problem try to avoid the solution of nonlinear systems. Among the implemented algorithms, worth special attention are the Lagutinski's method of determinants and the method proposed by Jacques–Arthur Weil in 1985 based on power series expansion [12].

2. The bounded de Beaune problem and Lagutinski's method of determinants

Lagutinski's method allows searching for particular and general integrals of ring derivations of sufficiently general form. An up-to-date presentation of this method for the case of the $\mathbb{C}[x, y]$ ring is given in [13, 14], and the general case is considered in [15]. For convenience of reference we present here a brief description of the method.

Let R be a ring with derivation D and field of constants k. Consider k to be an arbitrary field of characteristic zero and $\mathbb{Q} \in k$. Let us call a general integral of this derivation a pair of elements ψ_1, ψ_2 linearly independent over the field k, satisfying the equality

$$\psi_1 D\psi_2 = \psi_2 D\psi_1. \tag{3}$$

If the ring R is integral, then the derivation is naturally continued on its field of quotients, and the fraction ψ_1/ψ_2 satisfies the equation

$$D(\psi_1/\psi_2) = 0.$$

We will deal with rings where a basis can be introduced in the following sense.

Definition 1. A countable ordered set B of elements m_j of a ring R will be called a basis of the ring if

1) any element of the ring R can be presented as a linear combination of a finite number of elements of the set B with constant coefficients;
2) a product of any two elements of the set B belongs to B, and follows strictly after both efficients, i.e., $m_i m_j = m_n$ and n is strictly greater than i and j.

Let us introduce the ordering relationship in the basis, i.e., the inequality $m_i < m_j$ means that $i < j$ and assume that the notation $u = o(m_i)$ means that the representation of the element u of the ring R in the form of a linear combination of basis elements contains the basis elements whose numbers are strictly larger than i. If $u = am_i + o(m_i)$, $a \neq 0$, then the addend am_i will be called the lowest term in u.

In contrast to the common agreement, we call the number of the greatest basis term entering the decomposition of an element u in the basis an order of this element.

Example 2. In the ring $R = \mathbb{Q}[x, y]$ a system of various monomials may be taken to be a basis by accepting the glex-ordering:

$$1, y, x, y^2, xy, x^2, y^3, y^2x, yx^2, x^3, \ldots$$

Below this basis will be referred to as glex-basis. In this case, for example,

$$y^2 + xy + 3x^3 = y^2 + o(y^2),$$

and the order of this element equals 10.

The calculations of integrals is closely related to Lagutinski's determinants.

Definition 2. Compose an infinite matrix with the first row

$$m_1, m_2, \ldots,$$

the second row being the first derivative of the first one,

$$Dm_1, Dm_2, \ldots,$$

the third row being the second derivative of the first one,

$$D^2m_1, D^2m_2, \ldots,$$

and so on to infinity. A determinant of the corner minor of the n-th order of this matrix, i.e.,

$$\det \begin{pmatrix} m_1 & m_2 & \ldots & m_n \\ Dm_1 & Dm_2 & \ldots & Dm_n \\ \vdots & \vdots & \ddots & \vdots \\ D^{n-1}m_1 & D^{n-1}m_2 & \ldots, & D^{n-1}m_n \end{pmatrix} \tag{4}$$

will be denoted by Δ_n and called Lagutinski's determinant of the n-th order.

The following theorem provides a complete solution of the bounded de Beaune problem.

Theorem 1 (by M. N. Lagutinski). *Let R be a ring of polynomials.*

1. *A general integral exists then and only then, when all Lagutinski's determinants of sufficiently high order are equal to zero.*
2. *A general integral of the order N exists then and only then, when $\Delta_N = 0$; in this case the integral can be calculated as a ratio of the corresponding minors of this determinant.*

The proof of Lagutinski's theorem and the rule of choosing minors to construct integrals is given in [15].

Remark 1. *From this theorem, in particular, it follows that finding a rational integral does not require the field extension. If p and q belong to $\mathbb{Q}[x, y]$ and there is an integral in $\mathbb{C}(x, y)$, then applying this theorem at $k = \mathbb{C}$, we see that for a certain N $\Delta_N = 0$. The calculation of Lagutinski's determinants does not lead beyond the field $\mathbb{Q}$. Therefor, applying this theorem at $k = \mathbb{Q}$, we arrive at the existence of an integral in the field $\mathbb{Q}(x, y)$. For this reason, below we mean the integral of an equation with integer coefficients to be an element of $\mathbb{Q}(x, y)$.*

Lagutinski's method agrees well with the concept of operating with rings, accepted in Sage [16]. We have written a package `Lagutinski` [17] in Sage, which allows calculation of Lagutinski's determinants and integrals in this environment. The package was presented in 2016 at a number of conferences on computer algebra [18–20]. Here we restrict ourselves to one example illustrating the application of this package. In more detail the technique of its application is described in [21].

Example 3. Let the Bernoulli differential equation be given,

$$y(x + 1)dy - (y^2 + x + 2)dx = 0,$$

which for certain possesses an algebraic integral. Let us find it using Lagutinski's method. For this purpose we specify in a usual manner the corresponding differential ring and its basis:

```
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: D=lambda phi: y*(x+1)*diff(phi,x)+(y^2+x+2)*diff(phi,y)
sage: B= sorted(((1+x+y)^5).monomials(),reverse=0)
```

and load our package

```
sage: load(’’lagutinski.sage’’)
None
```

Now we can calculate Lagutinski's determinants, e.g.,

```
sage: lagutinski_det(2,B)
y^2 + x + 2
sage: lagutinski_det(3,B)
x^3 + x*y^2 + 5*x^2 + y^2 + 8*x + 4
```

Let us find that of the determinants, which equals zero:

```
sage: lagutinski_det(5,B)==0
False
sage: lagutinski_det(6,B)==0
True
```

Since $\Delta_5 \neq 0$, and $\Delta_6 = 0$, the integral will be:

```
sage: lagutinski_integral(6,B)
(-54*x^2 + 18*y^2 - 72*x)/(-18*y^2 - 36*x - 54)
```

Since the calculations are cumbersome, the first argument of this function should coincide with the smallest number of zero determinant.

The theory and its implementation are illustrated by Yu Ying by the examples taken from the book of problems by A. F. Filippov, the report is published in [22]. The numerical experiments carried out show that Lagutinski's method practically allows fast and resource-saving detection of the presence of a rational integral. However, the method requires considerable computational costs for the calculation of this integral. Note that the problem of determining the boundary for the integral order, always discussed in theory, appeared insignificant in practice, since there were no differential equations in the book of problems, whose integral curves had the order of 10 or higher.

3. Necessary conditions for the existence of an integral of contracting derivation

In application to a non-bounded de Beaune problem the Lagutinski method yields a sequence of determinants $\Delta_1, \Delta_2, \dots$.

According to the theorem 1 this sequence is finite then and only then when an integral in $k(x, y)$ exists. However, its condition cannot be checked constructively, moreover, the calculation of determinants of the order of $20 \div 30$ already requires considerable computational costs. Therefore, it is important to transform this statement into a necessary condition of the integral existence, at least for some classes of derivations.

Definition 3. A derivation D of the ring A will be called contracting, if such basis $B = \{m_1, m_2, \dots\}$ exists in which

$$Dm_i = c_i m_i + o(m_i). \tag{5}$$

Any basis, in which the differentiation operation satisfies the conditions 5, will be called a contracted derivation of D, c_i will be called indices of contraction in the basis B.

Generally, there can be several contracting bases, and the indices of contraction n then can be different. The possibility of applying the integral existence criteria presented below essentially depend on the possibility to choose a basis that contracts a given derivation.

Remark 2. *The proposed name refers to the theory of contracting operators in Banach spaces. In the present case, of course, there is no norm, but the basis specifies a certain "topology", and the condition contained in the definition* *indicates the fact that the derivation D transforms the basis element m_i into the element Dm_i, which is a linear combination of basis elements whose numbers are greater than i.*

Example 4. In the ring $R = \mathbb{Q}[x, y]$ the derivation

$$D = (ay + cx + \dots)\frac{\partial}{\partial y} - (bx + \dots)\frac{\partial}{\partial x}, \tag{6}$$

is contracting with respect to glex-basis $B = \{1,\, y,\, x,\, y^2,\, yx,\, x^2, \dots\}$.
Indeed,

$$D(y^n x^m) = n(ay + cx + \dots)y^{n-1}x^m - m(bx + \dots)y^n x^{m-1} = \\ = (an - mb)y^n x^m + o(y^n x^m).$$

The numbers $an - mb$ that appeared here are indices of contraction.

Theorem 2 (necessary criterion for existence of general integrals). *A contracting derivation of a polynomial ring R admits a general integral only if among the indices of contraction there are equal ones.*

This simple criterion follows from theorem 1 using the following lemma.

Lemma 1. *Let the derivation D be contracting, then in a suitable basis*

$$\Delta_n = W(c_1, c_2, \dots, c_n)\prod_{i=1}^{n} m_i + o\left(\prod_{i=1}^{n} m_i\right),$$

where W is a Vandermonde determinant.

Proof. In a suitable basis

$$Dm_i = c_i m_i + o(m_i),$$

from where

$$Do(m_i) = o(m_i)$$

and further

$$D^m m_i = c_i^m m_i + o(m_i).$$

The Lagutinski determinant Δ_n is formed by linear combinations of the products

$$D^{i_1} m_1 D^{i_2} m_2 \cdots = (c_1^{i_1} c_2^{i_2} \dots)\prod_{i=1}^{n} m_i + o\left(\prod_{i=1}^{n} m_i\right),$$

and, therefore, is a sum of the expression

$$\sum_{i_1, i_2, \dots} (-1)^{\sigma(i_1, i_2, \dots)} c_1^{i_1} c_2^{i_2} \dots \prod_{i=1}^{n} m_i + o\left(\prod_{i=1}^{n} m_i\right)$$

and higher-order terms. In the expression written out it is easy to recognize a Vandermonde determinant

$$W(c_1, \dots, c_n) = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ c_1 & c_2 & \dots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{n-1} & c_2^{n-1} & \dots & c_n^{n-1} \end{pmatrix}.$$

Example 5. The derivation

$$D = (x + x^4 y)\frac{\partial}{\partial x} + (x + y)\frac{\partial}{\partial y}$$

of the ring $\mathbb{Q}[x, y]$ is contracting, since in the glex-basis

$$B = \{1, y, x, y^2, xy, x^2, \dots\}$$

is true

$$D(x^n y^m) = n x^n y^m + (x+y) m x^n y^{m-1} + o(x^n y^m) = (n+m) x^n y^m + o(x^n y^m).$$

The indices of contraction form a sequence $0, 1, 1, 2, \dots$, in which equal elements are present. Therefore

$$\Delta_2 = W(0, 1)y + o(y) = y + o(y),$$

and then we obtain only

$$\Delta_n = W(0, 1, 1, \dots) \prod_{i=1}^{n} m_i + o\left(\prod_{i=1}^{n} m_i\right) = o\left(\prod_{i=1}^{n} m_i\right).$$

For small orders n the validity of this formula is easily checked by direct calculation:

```
sage: D=lambda phi: (x+x^4*y)*diff(phi,x)+(x+y)*diff(phi,y)
sage: prod(B[:2])
y
sage: sorted(lagutinski_det(2,B).monomials(),reverse=0)
[y, x]
sage: prod(B[:3])
x*y
sage: sorted(lagutinski_det(3,B).monomials(),reverse=0)
[x^2, x^4*y^2, x^5*y, x^6, x^7*y^3, x^8*y^2]
sage: prod(B[:4])
x*y^3
sage: sorted(lagutinski_det(4,B).monomials(),reverse=0)
[x^2*y^2, x^3*y, x^4, x^4*y^4, x^5*y^3, x^6*y^2, x^7*y, x^8,
x^7*y^5, x^8*y^4, x^9*y^3, x^10*y^2, x^11*y, x^10*y^6,
x^11*y^5, x^12*y^4, x^13*y^3]
```

4. Necessary conditions for the existence of a rational integral of the Briot–Bouquet equation

The theorem 2 is convenient for applying to the problem of finding an algebraic integral of the differential equation

$$(ay + cx + \dots)dx + (bx + \dots)dy = 0, \tag{7}$$

which we, following E. Ains [23, n. 12.6], will refer to as the Briot–Bouquet equation.

Remark 3. *Equation* (7) *possesses a number of unexpected analytical properties and for a long time attracts the attention of researchers. The initial problem*

$$\begin{cases} (ay + cx + \dots)dx + (bx + \dots)dy = 0, \\ y|_{x=0} = 0 \end{cases}$$

does not satisfy the conditions of the Cauchy theorem. Nevertheless, in 1856 Briot and Bouquet [23, n. 12.6], has proved that at $a/b \notin \mathbb{Z}$ this problem admits a unique solutions holomorphic in the vicinity of zero. The question of whether the initial problem admits other solutions having a singularity at zero, was the subject of research by Briot and Bouquet, Picard and Poincaré [24, n. 426].

An integral of the equation (7) is also an integral of the derivation

$$D = (ay + cx + \dots)\frac{\partial}{\partial y} - (bx + \dots)\frac{\partial}{\partial x}, \tag{8}$$

which as it has been shown in the example 4, contracts the glex-basis

$$B = \{1, y, x, y^2, xy, x^2, \dots\}.$$

From here, as a consequence of theorem 2, immediately follows:

Theorem 3 (about the Briot–Bouquet equation). *The differential Briot–Bouquet equation* (7) *can have a rational integral in $k(x, y)$ only if a and b are linearly dependent over the field $\mathbb{Q}$.*

Proof. Applying the derivation (8) to a monomial, we get

$$Dx^n y^m = (ay + cx + \dots)\frac{\partial x^n y^m}{\partial y} - (bx + \dots)\frac{\partial x^n y^m}{\partial x} = \\ = (ma - nb)x^n y^m + o(x^n y^m).$$

If there are no integer relations between a and b, then among the indices of contraction $ma - nb$ there are no equal ones, so that according to theorem 2 this derivation does not admit general integrals. □

Example 6. The general solution of the linear equation

$$(ay + cx)dx + bxdy = 0$$

is easy to write out

$$x^{a/b}\left(y + \frac{cy}{b+a}\right) = C,$$

where C is the integration constant. Whether the written integral is algebraic or not, depends on whether the ratio a/b is a rational number or not, which completely agrees with the proved lemma.

Example 7. According to the proved theorem the equation

$$(ay + cx)dx + (bx + xy)dy = 0.$$

has no algebraic integral at arbitrary a and b.

5. Necessary conditions for the existence of a rational integral concerning a fixed singular point

It is easily seen that the point $(0, 0)$ is a fixed singular point of the differential equation (7). Recall that the Cauchy theorem is applicable to all points of the xy-plane except those in which the polynomials p and q from $\mathbb{C}[x, y]$ simultaneously turn into zero. These points are called fixed singular points of the differential equation [10]. If we put the origin of the coordinate system into a fixed singular point, then

$$pdx + qdy = (a_{11}x + a_{12}y + \dots)dx + (a_{21}x + a_{22}y + \dots)dy,$$

where ... denote the terms of the order higher than the first one. The coefficient a_{22} prevents the application of theorem 3, however, it is easy to get rid of it by a linear change of variables.

Theorem 4. *Let neither p, nor q be reducible to a constant and the field of constants k is algebraically closed. Then to any fixed singular point (x_0, y_0) of the differential equation we can relate a new system of coordinates*

$$\begin{cases} x = x_0 + \xi + \alpha\eta, & \alpha \in k \\ y = y_0 + \eta, \end{cases} \tag{9}$$

in which this differential equation takes the form of Briot–Bouquet equation, i.e.,

$$(a\eta + c\xi + \dots)d\xi + (b\xi + \dots)d\eta,$$

where ... denotes the higher-order terms.

Proof. Since the field k is algebraically closed, the curves $p(x, y) = 0$ and $q(x, y) = 0$ intersect at some points of the xy-plane. Let us denote one on these points as (x_0, y_0) and put the origin of coordinates into this point. Then

$$pdx + qdy = (a_{11}x + a_{12}y + \dots)dx + (a_{21}x + a_{22}y + \dots)dy,$$

where ... denotes higher-order terms. The differential equation $pdx + qdy = 0$ corresponds to the derivation

$$D = (a_{11}x + a_{12}y + \dots)\frac{\partial}{\partial y} - (a_{21}x + a_{22}y + \dots)\frac{\partial}{\partial x}.$$

If $a_{22} \neq 0$, then it can be eliminated by a linear transformation

$$\begin{cases} x = \xi + \alpha\eta, & \alpha \in k, \\ y = \eta. \end{cases}$$

Under this transformation the form changes as follows:

$$pdx + qdy = (\dots)d\xi + [(a_{11}(\xi + \alpha\eta) + a_{12}\eta)\alpha + a_{21}(\xi + \alpha\eta) + a_{22}\eta + \dots]d\eta.$$

Equating the coefficient at $\eta d\eta$ to zero, we arrive at the quadratic equation

$$a_{11}\alpha^2 + (a_{12} + a_{21})\alpha + a_{22} = 0$$

for finding the parameter α. Since the field k is algebraically closed, this quadratic equation has roots in k, and for such a choice of the parameter the expression will get the desired form

$$pdx + qdy = (a\eta + c\xi + \dots)d\xi + (b\xi + \dots)d\eta.$$

Collecting the results of theorems 3 and 4 together, we get the following algorithm that allows clarifying whether the given differential equation (1) has a rational integral in the field $k(x, y)$:

1) find the fixed singular point (x_0, y_0);
2) execute a linear transformation, containing the parameter α, in the form

$$pdx + qdy = (a_{11}\xi + a_{12}\eta + \dots)d\xi + (a_{21}\xi + a_{22}\eta + \dots)d\eta;$$

3) determine the value of the parameter α from the quadratic equation $a_{22} = 0$;
4) check whether for such value of α the coefficients a_{12} and a_{21} are linearly dependent over $\mathbb{Q}$.

If yes, they are linearly dependent, then the differential equation can admit a rational integral, otherwise it does not exist. It is worth noting that the formulated criterion is necessary, but not sufficient.

Our Lagutinski package includes the function `lagutinski_ab`, which for specified p and q returns `true`, if at the first fixed singular point the above quantities are linearly dependent.

Example 8. For checking, let us begin with the linear equation

$$(x + y)dx + xdy = 0,$$

the general solution of which is expressed as

$$y(x) = -\frac{x}{2} + \frac{C}{x}.$$

We have:

```
sage: x,y=var('x,y')
sage: lagutinski_ab(x+y,x)
True
```

Example 9. Maple cannot make any definite conclusion about the equation

$$(2-x^2-y^2)dx+(x-y)dy=0.$$

The application of our criterion yields

```
sage: x,y=var('x,y')
sage: lagutinski_ab(2-x^2-y^2,x-y)
False
```

Therefore, this equation does not admit a rational integral in the field $\mathbb{C}(x,y)$.

It is well known that an arbitrary differential equation (1) cannot be integrated in elementary functions. The proposed algorithm specifies the "degeneracies" that should occur with the coefficients p and q of the differential equation considered to make it integrable in such functions. If the polynomials p, q belong to $\mathbb{Q}[x,y]$, then the application of the described algorithm introduces algebraic numbers twice: first, in finding the fixed singular points and, second, in searching for the parameter α. Therefore, generally the ratio of the coefficients a_{12} and a_{21} appears to be an algebraic number, so that the equation does not admit an algebraic integral even in $\mathbb{C}(x,y)$.

Remark 4. *It is natural to draw an analogy here with the integration of rational functions: in the general case, the denominator of a rational function has simple zeros, and the integral of such a function consists of logarithmic terms; the integral will be rational only in the exotic case when all the singularities are multiple.*

6. Differential equations with symbolic parameters

The theorem 3 is seen useless in the case, when the coefficients of Briot–Bouquet equation belong to the field $\mathbb{Q}$. Actually, theorem 2 provides a convenient criterion of unsolvability when the considered equation contains indefinite parameters $a, b, \dots$, in other words, when as the field k we consider the field $\mathbb{Q}(a,b,\dots)$, generated by the variables $a, b, \dots$ algebraically independent over $\mathbb{Q}$. With their appearance the problem of finding an algebraic integral is separated into two problems:

— to clarify whether the differential equation admits a rational integral in the field $k(x,y)$, i.e., "in the general case";
— to find particular values of the parameters $a, b, \dots$ in $\mathbb{C}$, for which the differential equation admits a rational integral in the field $\mathbb{C}(x,y)$.

The first problem for the equation (7) was completely solved by theorem 3: this equation has no rational integral in the general case.

Now let us proceed to the second problem. Without loss of generality, we can assume that p and q are polynomials with respect to x, y and all parameters $a, b, \dots$; for clarity let us consider the set of complex values of the parameters $a, b, \dots$ as a point in a finite-dimensional affine space A over the field $\mathbb{C}$. Accepting this agreement and using common notations of algebraic geometry [25], the theorem 1 for the field $R = Q[x, y, a, b, \dots]$ can be reformulated in the following way.

Theorem 5. *Let the coefficients Δ_n of monomials $x^n y^m$ generate an ideal J_n of the ring $\mathbb{Q}[a, b, \dots]$. The set of points $(a, b, \dots)$ of the affine space A, for which the differential equation*

$$pdx + qdy = 0, \quad p, q \in \mathbb{Q}[x, y, a, b, \dots],$$

admits a rational integral from $\mathbb{Q}(x, y)$, whose order does not exceed, is an algebraic affine set $Z(J_n)$ in A.

Proof. If the point $(a, b, \dots)$ belongs to $Z(J_n)$, then $\Delta_n(x, y, a, b, \dots)$ at such values of parameters $a, b, \dots$ indentically turns into zero, and due to the Lagutinski theorem 1 the differential equation admits a rational integral. Conversely, if for some values of the parameters $a, b, \dots$ the differential equation admits a rational integral of the order n, then the Lagutinski determinant of the same order turns into zero identically and, therefore, $(a, b, \dots)$ belongs to $Z(J_n)$. □

Generally, the set $Z(J_n)$ can be empty or reducible.

Example 10. Consider again the linear equation

$$(ay + cx)dx + bxdy = 0.$$

Let us specify the appropriate ring, derivation, and basis:

```
sage: R.<x,y,a,b,c> = PolynomialRing(QQ, 5)
sage: D=lambda phi: (a*y+c*x)*diff(phi,y) -b*x*diff(phi,x)
sage: B= sorted(((1+x+y)^30).monomials(),reverse=0)
```

Calculate the Lagutinski determinants:

```
sage: lagutinski_det(2,B).factor()
y*a + x*c
sage: lagutinski_det(3,B).factor()
b * a * x * (y*a + y*b + x*c)
sage: lagutinski_det(4,B).factor()
(-2) * b * a * x * (y*a + x*c) * (y*a + y*b + x*c) * (-
2*y*a^2 - y*a*b - 2*x*a*c + 2*x*b*c)
```

In the three-dimensional affine space A the set $Z(J_2)$ represents a straight line $\{a = 0, c = 0\}$, the sets $Z(J_3), Z(J_4)$ represent a union of two planes $\{a = 0\} \cup \{b = 0\}$, and so on.

According to the theorem 5 the values of the parameters $a, b, \dots$, for which the differential equation

$$pdx + qdy = 0, \quad p, q \in \mathbb{Q}[x, y, a, b, \dots]$$

admits a rational integral of any order in $\mathbb{C}(x, y)$ for the set $\cup Z(J_n)$. It could be expected that this set is also algebraic, as it usually happens in algebraic problems. However, this is not true.

Example 11. The differential equation from the example 6 has a rational integral then and only then, when the ratio a/b is a rational number or when $b = 0$. Therefore $\cup Z(J_n)$ represents a union of various planes

$$na + mb = 0, \quad n, m \in \mathbb{Z}$$

in the three-dimensional affine space A.

Now let us reformulate the theorem 3 in these terms.

Theorem 6. *The projection onto the plane ab of a set of all values of the parameters $a, b, \dots$, at which the differential equation* (7) *admits a rational integral in $\mathbb{C}(x, y)$, is a union of a certain number of straight lines of the form*

$$na + mb = 0, \quad n, m \in \mathbb{Z}$$

and points.

Proof. According to the theorem 5 the set of all points of affine space A, at which the differential equation admits an integral from $\mathbb{C}(x, y)$, is a sum of algebraic affine sets and, therefore, represents a union of irreducible affine manifolds. And according to the theorem 3 a projection of this set onto the plane ab is formed by points that are linearly dependent over $\mathbb{Q}$.

This projection cannot coincide with the entire plane, therefore, it can be decomposed into irreducible lines and points. Assume, in contradiction to the theorem, that among these lines there is an irreducible line C of the order r, different from straight lines

$$na + mb = 0, \quad n, m \in \mathbb{Z}.$$

According to the theorem 3 for any point $(a, b) \in C$ of this curve it is possible to specify one and only one such pair of mutually simple integer numbers (n, m) that

$$na + mb = 0, \quad m \geqslant 0.$$

From a geometric point of view this means that any point $(a, b) \in C$ corresponds to the point (n, m) of a projective straight line $P^1_{\mathbb{Q}}$, i.e., we get a mapping

$$f : C \to P^1_{\mathbb{Q}}.$$

The prototype of the point (n, m) is the set of points (a, b) of the line C at which the equality

$$na + mb = 0,$$

i.e., the points of intersection of the straight line $na + mb = 0$ and the line C in the plane $A^2_{\mathbb{C}}$. By Bézout's theorem, there are exactly r such points, therefore, there is a $(1, r)$-correspondence between the affine line C over $\mathbb{C}$ and the projective straight line P^1 over $\mathbb{Q}$. As soon as the set $\mathbb{Q}$ is countable and the set of points of the algebraic line over $\mathbb{C}$ is uncountable, the above is impossible. Hence, the projection is a union of straight lines

$$na + mb = 0, \quad n, m \in \mathbb{Z}.$$

and points. □

As shown by example 11, the projection of a set of parameter sets $a, b, \dots$, at which the differential equation admits an algebraic integral can be a union of countable sets of affine manifolds, projected into a family of straight lines

$$na + mb = 0, \quad n, m \in \mathbb{Z}.$$

If, as Lagutinski hoped for, it would be possible to replace an infinite sequence of determinants $\Delta_2(x, y, a, b, \dots)$, $\Delta_3(x, y, a, b, \dots), \dots$ with a finite set of conditions, then this set would be an affine set. Thus, the appearance of the infinite sequence is not a defect of the Lagutinski method, it indicates the non-algebraic component of the theory of integration of differential equations in algebraic functions. Thus, the bounded de Beaune problem is completely solved by the Lagutinski method, and the unbounded de Beaune problem with parameters inevitably introduces non-algebraic sets and therefore, it does not admit a purely algebraic method of solution.

Conclusion

To summarize, let us list the main results of our consideration:

— Lagutinski's method allows solving the bounded de Beaune problem 2 using a finite number of operations, its implementation in Sage faces but one difficulty: with the growth of the boundary N the calculation of determinants requires more and more computer resources. The calculations can be made faster by choosing a suitable basis; in contracted bases the calculations are considerably more rapid (see lemma 1).

— For the unbounded problem 1 it appears possible to derive from Lagutinski theorem the necessary and easily checked conditions of existence of a rational integral. These criteria are applicable also in the cases, when the standard approaches implemented in Maple yield no definite information, see example 9.

— The above criterion appears to be rather useful for that problems with parameters, when for a given differential equation, containing indefinite parameters, one has to choose their values in a way providing the particular differential equation to admit an algebraic integral. This case clearly demonstrates the reasons why the full solution of an unbounded de Beaune problem is impossible: the desired set of the parameter values is not always an algebraic set.

Acknowledgments

The Lagutinski package was developed by M.D. Malykh and Yu Ying within the frameworks of a project, supported by the Russian Foundation for basic Research (grant No. 18-51-18005). The studies of necessary conditions for the existence of a contracting derivation integral were carried out by Yu Ying under the guidance of L. A. Sevastianov and supported by the Program RUDN 5-100. The calculations presented in the paper were performed in the computer algebra system Sage [16].

References

1. R. Descartes, Œuvres, Vol. 2, Léopold Cerf, Paris, 1898.
2. H. Poincaré, Œuvres, Vol. 3, Gautier, Paris, 1934.
3. M. N. Lagutinski, Applying polar operations to the integration of ordinary differential equations in finite form [Prilozhenie poljarnyh operacij k integrirovaniju obyknovennyh differencial'nyh uravnenij v konechnom vide], Soobshh. Har'kov. matem. obshh. Vtoraja serija 12 (1911) 111–243, in Russian.
   URL `http://mi.mathnet.ru/khmo117`
4. M. N. Lagutinski, On some polynomials and their relationship to algebraic integration of ordinary differential algebraic equations [O nekotoryh polinomah i svjazi ih s algebraicheskim integrirovaniem obyknovennyh differencial'nyh algebraicheskih uravnenij], Soobshh. Har'kov. matem. obshh. Vtoraja serija 13 (1912) 200–224, in Russian.
   URL `http://mi.mathnet.ru/khmo104`
5. V. A. Dobrovol'skij, N. V. Lokot', S. J.-M., Mikhail Nikolaevich Lagutinskii (1871–1915): un mathématicien méconnu, Historia Mathematica 25 (1998) 245–64.
6. A. J. Maciejewski, J.-M. Strelcyn, On the algebraic non-integrability of the Halphen system, Physics Letters A 201 (1995). doi:10.1016/0375-9601(95)00285-B.
7. Ngoc Thieu Vo, F. Winkler, Algebraic general solutions of first order algebraic ODEs, Vol. 9301, Springer, Cham, 2015, pp. 479–492. doi:10.1007/978-3-319-24021-3_35.
8. M. D. Malykh, On integration of the first order differential equations in finite terms, IOP Conf. Series: Journal of Physics: Conf. Series 788, article number 012026 (2017). doi:10.1088/1742-6596/788/1/012026.
9. E. S. Cheb-Terrab, Computer algebra solving of first order ODEs, Computer physics communications 101 (1997) 254–268. doi:10.1016/S0010-4655(97)00018-0.
10. W. W. Golubew, Vorlesungen über Differentialgleichungen im Komplexen, Deutscher Verlag der Wissenschaften, Berlin, 1958.
11. Fr. Severi, Lezioni di geometria algebrica, Angelo Graghi, Padova, 1908.
12. A. Bostan, G. Chéze, T. Cluzeau, J.-A. Weil, Efficient Algorithms for Computing Rational First Integrals and Darboux Polynomials of Planar Polynomial Vector Fields, Mathematics of Computation 85 (2016) 1393–1425. doi:10.1090/mcom/3007.
13. C. Christopher, J. Llibre, J. Vitório Pereira, Multiplicity of invariant algebraic curves in polynomial vector fields, Pacific Journal of Mathematics 229 (1) (2007) 63–117. doi:10.2140/pjm.2007.229.63.
14. G. Chèze, Computation of Darboux polynomials and rational first integrals with bounded degree in polynomial time, Journal of Complexity 27 (2) (2011) 246–262. doi:10.1016/j.jco.2010.10.004.
15. M. D. Malykh, On the computation of the rational integrals of systems of ordinary differential equations by Lagutinski's method [Ob otyskanii ratsional'nykh integralov sistem obyknovennykh differentsial'nykh uravneniy po metodu M.N. Lagutinskogo], Bulletin of NRNU MEPhI [Vestnik Natsional'nogo issledovatel'skogo yadernogo universiteta "MIFI"] 5 (24) (2016) 327–336, in Russian. doi:10.1134/S2304487X16030068.

16. The Sage Developers, SageMath, the Sage Mathematics Software System (Version 7.4) (2016).
URL `https://www.sagemath.org`
17. M. D. Malykh, Lagutinski.sage, ver. 1.5., RUDN University (2016).
URL `http://malykhmd.neocities.org`
18. M. D. Malykh, On M.N. Lagutinski method for integration of ordinary differential equations, in: International conference "Polynomial Computer Algebra'2016", 2016, pp. 57–58.
URL `http://pca.pdmi.ras.ru/2016/pca2016book.pdf`
19. M. D. Malykh, On the integration of ordinary differential equations [Ob integrirovanii obyknovennyh differencial'nyh uravnenij], in: Computer algebra. Proceedings of the international conference, June 29 – July 2, 2016, Moscow, Russia, 2016, pp. 25–29, in Russian.
20. M. D. Malykh, On the integration of first-order differential equations in finite form [Ob integrirovanii differencial'nyh uravnenij pervogo porjadka v konechnom vide], in: Fifth International Conference on Problems of Mathematical and Theoretical Physics and Mathematical Modelling. Moscow, April 5–7, 2016. Collection of reports, 2016, pp. 81–82, in Russian.
21. M. D. Malykh, On application of M. N. Lagutinski method to integration of differential equations in symbolic form. Part 1 [O yavnom atribute M.N. Lagutinskogo k integrirovaniyu differentsial'nykh uravneniy 1-go poryadka. Chast' 1. Otyskaniye algebraicheskikh integralov], RUDN Journal of Mathematics, Information Sciences and Physics 25 (2) (2017) 103–112, in Russian. doi:10.22363/2312-9735-2017-25-2-103-112.
22. M. D. Malykh, Yu Ying, The Method of finding algebraic integral for first-order differential equations [Metodika otyskaniya algebraicheskikh integralov differentsial'nykh uravneniy pervogo poryadka], RUDN Journal of Mathematics, Information Sciences and Physics 26 (3) (2018) 285–291, in Russian. doi:10.22363/2312-9735-2018-26-3-285-291.
23. E. L. Ince, Ordinary differential equations, Courier Corporation, 1956.
24. É. Goursat, Cours d'analyse mathématique, Vol. 2, Gauthier-Villars, Paris, 1925.
25. R. Hartshorne, Algebraic geometry, Springer, 1977.

For citation:

M. D. Malykh, L. A. Sevastianov, Yu. Ying, On algebraic integrals of a differential equation, Discrete and Continuous Models and Applied Computational Science 27 (2) (2019) 105–123. DOI: 10.22363/2658-4670-2019-27-2-105-123.

Information about the authors:

Mikhail D. Malykh (Russian Federation) — Candidate of Physical and Mathematical Sciences, assistant professor of Department of Applied Probability and Informatics of Peoples' Friendship University of Russia (RUDN University) (e-mail: `malykh-md@rudn.ru`, phone: +7(495)9550927, ORCID: https://orcid.org/0000-0001-6541-6603, ResearcherID: P-8123-2016, Scopus Author ID: 6602318510)

Yu Ying (China) — postgraduate student of Department of Applied Probability and Informatics of Peoples' Friendship University of Russia (RUDN University); assistant professor of Department of Algebra and Geometry, Kaili University (e-mail: yingy6165@gmail.com, phone: +7(495)9550927, ORCID: https://orcid.org/0000-0002-4105-2566, ResearcherID: AAC-8344-2019, Scopus Author ID: 57208127921)

Leonid A. Sevastianov (Russian Federation) — professor, Doctor of Physical and Mathematical Sciences, professor of Department of Applied Probability and Informatics of Peoples' Friendship University of Russia (RUDN University) (e-mail: sevastianov-la@rudn.ru, phone: +7(495)955-07-83, ORCID: https://orcid.org/0000-0002-1856-4643, ResearcherID: B-8497-2016, Scopus Author ID: 8783969400)