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Results on Simply-Continuous Functions

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In this paper, we introduce new classes of simply continuous functions as generalization of continuous function. We obtain their characterizations, their basic properties and their relationships with other forms of generalized continuous functions between topological spaces.

Key words and phrases: simply-opened set, simply-closed set, δ-set, semi-g-regular space, almost continuous function.

1. Introduction

Let (X, τ) be a topological space. For a subset S of X, the closure, the interior, and the complement of S with respect to (X, τ) is denoted by $cl\, S$, $int\, S$, and S^c, respectively. Any function is assumed to be mapping between two topological spaces.

Recently there has been some interest in the notion of a simply-opened subset of topological space (X, τ). According to Neubrunnovia [1] a subset S of a space (X, τ) is called simply-opened if it is $S = O \cup N$, where O is open and N is nowhere dense $(= nwd, int\,(cl\, N) = \Phi)$ subset of a space (X, τ).

In [2] Ganster, Reilly and Vamanmurthy showed that a subset S of a space (X, τ) is simply-opened if and only if it is intersection of semi-opened and semi-closed subsets of a space (X, τ).

A set B of a topological space (X, τ) is simply-closed if and only if B^c is simply-opened.

The class of all simply-opened subsets of the space (X, τ) will be denoted by $SMO(X)$.

Semi-opened subsets of a topological space were defined by Levine [3]. Recall that S is called semi-opened [3] if $S \subset cl(int\, S)$. A semi-closed set is a set, whose complement is semi-opened.

A subset S of a topological space (X, τ) is called $\delta - set$(see [4]) if $int(cl\, S) \subset cl(int\, S)$.

Intersection of all semi-closed sets containing A is called the semi-closed of A (see [5]) and is denoted by $scl(A)$. By a semi-clopen set we mean the set, which is semi-opened and semi-closed at the same time. Recall that a subset A of a space (X, τ) is called regular open if $A = int\, cl\, A$.

The concept of semi-reopening was introduced by D. Andrijevic (see [6] and [7]). These sets were called β — open sets. A subset S of a space X is called semi-preopen if $S \subset cl(int(cl(S)))$.

A subset A of a space X is said to be regular open (respectively regular closed)if $A = int(cl(A))$ (respectively $A = cl(int(A))$) (see [7]).

In [8] the definition of semi-g-regular topological space was introduced. A topological space (X, τ) is said to be semi-g-regular if for each sg-closed set A and each point $x \notin A$, there exist disjoint semi-opened sets $U, V \subset X$ such that $A \subset U$ and $x \in V$.

A subset A of a topological space (X, τ) is called semi-generalized closed (see [9]), briefly sg-closed, if $scl(A) \subset U$ whenever $A \subset U$ and U is a semi-opened.

A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be semi-continuous [3] (resp. irresolute [10], perfectly contra-irresolute [11], semi-precontinuous [12], almost continuous [13] (written as a.c. S), contra-irresolute [11]) if the inverse image of every open subset (resp. semi-opened, semi-opened, open, regular open, semi-opened) subset of (Y,σ) is semi-opened (resp. semi-opened, semi-clopen, semi-preopen, open, semi-closed) in (X,τ).

A function $f : (X,\tau) \longrightarrow (Y,\sigma)$ between two topological spaces is said to be pre-semi-opened [10] if and only if, the image of every semi-opened set in (X,τ) is semi-opened in (Y,σ).

A function $f : (X,\tau) \longrightarrow (Y,\sigma)$ between two topological spaces is said to be semi-homeomorphism [10] if and only if f is one-to-one, onto, irresolute, and pre-semi-opened.

Def 1. Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be a function. We say that f is a simply-continuous(resp. ws-continuous, ss-continuous if the inverse image of every open(resp. simply-opened, simply-opened) set in Y is simply-opened(resp. open, simply-opened) set in X.

We give now an example to show that the simply-continuous function appears to be not necessary continuous function.

Example 1. Let $f : \Re \longrightarrow \Re$ where $\Re$ is the real line with the usual topology and $f(x) = 1$ when $(0 \leqslant X < 1)$ and $f(x) = 0$ otherwise. It is very easy to see that f is simply-continuous but not continuous function.

2. Properties of simply-continuous, ss- continuous, ws-continuous functions

Lemma 1 (see [14]). *Every semi-opened subset of a space (X,τ) is a simply-opened set.*

Theorem 1. *Every perfectly contra-irresolute function is a simply-continuous function.*

Proof. Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be a perfectly contra-irresolute function and let O be an open subset of Y. Then by lemma 1 we have that O is simply-opened and since f is perfectly contra -irresolute function, f^{-1} (O) is semi-clopen subset of X $(f^{-1}(O) = f^{-1}(O) \cap f^{-1}(O))$ as intersection of semi-opened and semi-closed and so, f^{-1} (O) is simply-opened by the definition of simply-opened in [2]. $\square$

Theorem 2. *Every ws-continuous is a semi-precontinuous function.*

Proof. Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be a ws-continuous function and let$A$ be an open and so semi-opened subset of Y. By lemma 1 A is simply-opened subset of Y, so $f^{-1}(A)$ is open subset of X and so semi-opened, but it is very easy to see that, every semi-opened set is semi-preopen, so $f^{-1}(A)$ is semi-preopen. $\square$

The converse of above theorem need not be true by the following example:

Example 2. Let $X = \{a,b,c\}$, $\tau_1 = \{\Phi, a, b, a, b, X\}$, $Y = \{1,2,3,4\}$, $\tau_2 = \{\Phi, 1, 1, 2, 1, 2, 3, Y\}$. A function $f : (X,\tau_1) \longrightarrow (Y,\tau_2)$ is defined by $f(a) = 1$, $f(b) = 3$, $f(c) = 2$. Here $SPO(X) = \{\Phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, X\}$. Then f is semi-precontinuous. But f is not ws-continuous since $f^{-}1(\{1,2\}) = \{a,c\}$, which is not open in τ_1.

Theorem 3. *Every ws-continuous function is an almost continuous function.*

Proof. Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be a ws-continuous function and let A be a regular open subset of Y. Since every regular open set is open set, and so semi-opened subset of Y, by lemma 1, A is simply-opened subset of Y, so $f^{-1}(A)$ is an open subset of X.$\square$

Proposition 1 (see [15]). If X is a topological space and Y is a dense subspace of X, then for any $A \subset Y$, A is nowhere dense in Y if and only if A is nowhere dense in X.

Theorem 4. *Let (X, τ) be a topological space, (X_0, τ_0) be a subspace of (X, τ), $A \subset X_0 \subset X$ and $X_0 \in SMO(X)$. If $A \in SMO(X)$ then $A \in SMO(X_0)$.*

Proof. Since A is simply-opened in (X, τ) then $A = O \cup N$, where O is open and N is nowhere dense subsets of X, so $A = X_0 \cap A = X_0 \cap (O \cup N) = (X_0 \cap O) \cup (X_0 \cap N)$. But $(X_0 \cap O)$ and $(X_0 \cap N)$ are open and nowhere dense subsets of the space (X_0, T_0) respectively. Thus A is simply-opened in (X_0, T_0). □

Lemma 2 (see [14]). *Let S and O be subsets of a space (X, τ) such that S is a simply-opened set and O is an open set. Then $S \cap O$ is a simply-opened set.*

Theorem 5. *If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a simply-continuous function and X_0 is an open set in X, then the restriction $f/X_0 : X_0 \longrightarrow Y$ is simply-continuous.*

Proof. Since f is simply-continuous, for any open set V in Y, $f^{-1}(V)$ is a simply-opened set in X. Hence by lemma 2, we have that $f^{-1}(V) \cap X_0$ is a simply-opened set in X. Since X_0 is an open set, by theorem 8, $(f/X_0)^{-1}(V) = f^{-1}(V) \cap X_0$ is a simply-opened set in X_0, which implies that f/X_0 is simply-continuous. □

Theorem 6. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a function between two topological spaces. Then the following statements are equivalent:*
(1) f is strongly simply-continuous;
(2) if $x \in X$ and V is simply-opened in Y containing $f(x)$, then there exists U simply-opened in X containing x such that $f(U) \subset V$.

Proof ((1) $\Longrightarrow$ (2)). Let V be simply-opened in Y and $f(x) \in V$. Since f is ss-continuous, $f^{-1}(V)$ is simply-opened in X and $x \in f^{-1}(V)$. Put $U = f^{-1}(V)$. Then $x \in U$ and $f(U) = f(f^{-1}(V)) \subset V$.

((2) $\Longrightarrow$ (1)). Let V be simply-opened in Y and $x \in f^{-1}(V)$. Then $f(x) \in V$ therefore, there exists a U_x simply-opened in X such that $x \in U_x$ and $f(U_x) \subset V$, therefore $x \in U_x \subset f^{-1}(V)$. This implies that $f^{-1}(V)$ is union of simply-opened sets, hence $f^{-1}(V)$ is simply-opened in X so f is strongly simply-continuous. □

Theorem 7. *Let (X, τ) and (Y, σ) be two topological spaces. The function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is simply-continuous function if and only if, for every closed subset B of Y, $f^{-1}(B)$ is simply-closed in X.*

Proof. *Necessity.* If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is simply-continuous function, then for every open subset O of Y, $f^{-1}(O)$ is simply-opened in X. If B is any closed subset of Y, then B^c is open. Thus $f^{-1}(B^c)$ is simply-opened, but $f^{-1}(B^c) = (f^{-1}(B))^c$ so that $f^{-1}(B)$ is simply-closed.

Sufficiency. If for all closed subset B of Y, $f^{-1}(B)$ is simply-closed in X, and if O is any open subset of Y, then O^c is closed. Also, $f^{-1}(O^c) = (f^{-1}(O))^c$ is simply-closed. Thus $f^{-1}(O)$ is simply-opened. □

Theorem 8. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be two topological spaces. The one-to-one and onto function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is ws-continuous (resp. ss-continuous) if and only if, for every simply-closed subset B of Y, $f^{-1}(B)$ is closed(resp. simply-closed) in X.*

Proof. In the same way of proving theorem 7 we can proof this theorem. □

Lemma 3 (see [8]). *For a topological space (X, τ) the following statements are equivalent:*

(1) (X, τ) *is semi-g-regular;*
(2) every sg-closed set A *is an intersection of semi-clopen sets.*

By the upper Lemma we have the following proposition:

Proposition 2. A subset B of a semi-g-regular topological space is simply-opened if and only if it is sg-closed set.

Theorem 9. *Let* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *be a simply-continuous (resp. ws-simply continuous, ss-continuous) function between two semi-g-regular topological spaces. Then the inverse image of every sg-closed set is sg-closed set.*

Proof. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a simply-continuous (resp. ws-simply continuous, ss-continuous) function between two semi-g-regular topological spaces and let A be a sg-closed subset of Y and Y a semi-g-regular topological space, then, by Proposition 2, A is simply-opened subset of Y, so $f^{-1}(A)$ is simply-opened subset of X whatever f be. But X is semi-g-regular topological space, so $f^{-1}(A)$ is sg-closed set. □

Lemma 4 (see [11]). *Let* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *be a map. Then the following conditions are equivalent:*
(i) f *is contra-irresolute;*
(ii) the inverse image of each semi-closed set in Y *is semi-open in* X*.*

Theorem 10. *Every contra-irresolute function is simply-continuous.*

Proof. Let A be open in (Y, σ) where $f : (X, \tau) \longrightarrow (Y, \sigma)$ is contra-irresolute function, so A^c is closed and so semi-closed and by the contra-irresoluteness, lemma 4, of the function we have, $f^{-1}(A^c)$ to be semi-open and so, by lemma 1 is simply-closed in (X, τ), so we have that $(f^{-1}(A^c))^c = f^{-1}(A)$ is simply-opened. □

References

1. *Neubrunnovia.* On Transfinite Sequences of Certain Types of Functions // Acta Fac. Rer. Natur. Univ. Com. Math. — 1975. — Vol. 30. — Pp. 121–126.
2. *Ganster M., Reilly I. L., Vamanmurthy M. K.* Remarks on Locally Closed Sets // Mathematical Pannonica. — 1992. — Vol. 3(2). — Pp. 107–113.
3. *Levine N.* Semi-Open Sets and Semi-Continuity in Topological Space // Amer. Math. Monthly. — 1963. — Vol. 70. — Pp. 36–41.
4. *Chattopadhyay C., Bandyopadhyay C.* On Structure of $\delta - sets$. — 1991.
5. *Crossly S. G., Hildebrand S. K.* Semi-Closure // Texas J. Sci. — 1971. — Vol. 22. — Pp. 99–112.
6. *Andrijevic D.* Semi-Preopen Sets // Ibid. — 1986. — Pp. 24–32.
7. *Stone M. H.* Applications of the theory of Boolean Rings to General Topology // TAMS. — 1937. — Vol. 41. — Pp. 375–381.
8. *Ganster M., Jafari S., Navalagi G. B.* On Semi-g-Regular and Semi-g-Normal Spaces // Demonstration Math. — 2002. — Vol. 35. — Pp. 414–421.
9. *Bhattacharyya P., Lahiri B. K.* Semi-Generalized Closed Sets in Topology // Indian J. of Math. — 1987. — Vol. 29. — Pp. 375–382.
10. *Crossly S. G., Hildebrand S. K.* Semi-Topological Properties // Fund. Math. — 1974. — Vol. 74. — Pp. 233–254.
11. *Cueva M. C.* Weak and Strong Forms of Irresolute Maps // Internat. J. Math. Math. Sci.Vol. — 2000. — Vol. 23. — Pp. 253–259.
12. *Navalagi G. B.* Semi-Precontinuous Functions and Properties of Generalized Semi-Preclosed Sets in Topological Spaces // IJMMS Internat. J. Math. Math. Sci. — 2002. — Vol. 29. — Pp. 85–98.
13. *Singal M. K., Singal A. R.* Almost-Continuous Mappings // Yokohama Math. J. — 1968. — Vol. 15. — Pp. 63–73.

14. *Kljushin V. L., Jalal Hatem Hussein A. B.* On Simply-Open Sets // Bulletin of PFUR. Series "Mathematics. Informatics. Physics". — 2011. — Vol. 3. — Pp. 34–38.
15. *Maxim R. Burke A. W. M.* Models in which Every Nonmeager Set is Nonmeager in a Nowhere Dense Cantor Set // Canadian Journal of Mathematics. — 2005. — Vol. 57. — Pp. 1129–1144.

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Некоторые результаты о просто-непрерывных функциях

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В этой статье введены три класса функций, называемых соответственно просто-непрерывными, сильно просто-непрерывными и слабо просто-непрерывными как обобщения непрерывных функций. Получены их характеристики, основные свойства и описаны их связи с другими обобщениями непрерывных отображений топологических пространств.

Ключевые слова: простооткрытые множества, простозамкнутые множества, полу-g-регулярное пространство, почти непрерывная функция.