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Complexes of Localized States in Ac-Driven Nonlinear Schrödinger Equation and in Double Sine-Gordon Equation

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Complexes of localized states are numerically analyzed in two dynamical systems: directly driven nonlinear Schrödinger equation (NLS) and double sine-Gordon equation (2SG). Both systems have a wide range of physical applications. Our numerical approach is based on the numerical continuation with respect to the control parameters of the quiescent (stationary) solutions and stability and bifurcation analysis of the linearized eigenvalue problem. Multi-soliton complexes of the NLS equation are studied in the undamped and the weak damping regimes. We show that in the weak damping case the directly driven NLS equation holds stable and unstable multi-soliton complexes. The results are confirmed by means of direct numerical simulations of the time-dependent NLS equation. Properties of the multi-fluxon solutions of 2SG equation are studied depending on the parameter of the second harmonic. We show that the second harmonic changes properties and increases the complexity of coexisting static fluxons of 2SG equation. Results are discussed within the frame of the long Josephson junction model.

Key words and phrases: soliton, fluxon, Newtonian iteration, numerical continuation, stability.

1. Introduction

We study complexes of localized states in two dynamical systems: externally-driven nonlinear Schrödinger equation (NLS) and double sine-Gordon equation (2SG).

Both systems have undergone an extensive mathematical analysis because of their wide range of physical applications.

In both cases, our numerical approach is based on numerical continuation of stationary solutions of respective partial differential equations and linearized eigenvalue problems [1, 2]. Numerical continuation algorithm is described in [1, 3]. At each step of numerical continuation, the Newtonian iteration with the 4th order accuracy Numerov's discretization is utilized. Our aim is a numerical study of

(i) multi-soliton complexes of ac-driven NLS in the case of weak damping;
(ii) multi-fluxon solutions of 2SG depending on the second harmonic.

2. Complexes in the Ac-Driven, Weakly Damped NLS

We consider the nonlinear Schrödinger equation (NLS) driven by a constant external force

$$i\psi_t + \psi_{XX} + 2|\psi|^2\psi - \psi = -h - i\gamma\psi, \quad \psi_X(\pm\infty) = 0, \tag{1}$$

where $\gamma > 0$ and h are, respectively, parameters of the damping strength and the external driving. Two types of stationary soliton solutions of (1) (denoted ψ_- and ψ_+) are well investigated [4]. Strongly damped ($\gamma > 0.5$) stationary complexes of ψ_- and ψ_+ solitons were obtained in [1, 5]. Existence of stationary undamped complexes

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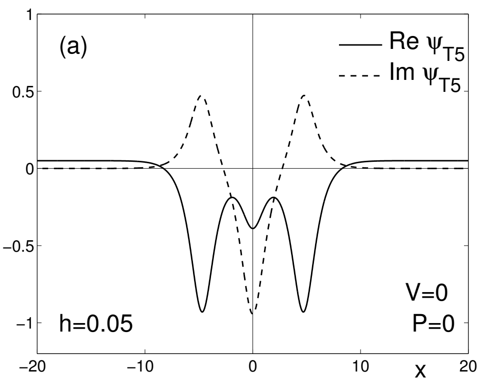
was proved in [5]. Also, travelling undamped waves and complexes were obtained in [5, 6]. Our aim is investigation of multi-soliton complexes in case of small damping ($0 < \gamma < 0.5$)

Stationary localized solutions of (1) are pathfollowed in γ for the fixed value h. Stability and bifurcations of stationary solutions of (1) are classified by means of numerical solution of the respective linearized eigenvalue problem, see [5]. At each step of the numerical continuation, we calculate the energy integral as follows:

$$E = \int \mathrm{d}x[|\psi_x|^2 + |\psi|^2 - |\psi|^4 - h(\psi + \psi^*) - |\psi_0|^2 + |\psi_0|^4 + h\,(\psi_0 + \psi_0^*)]\mathrm{d}x, \qquad \psi_0 = \psi(\pm\infty). \tag{2}$$

In [6], we obtained the undamped multi-soliton complex T5 (see Fig. 1a) which was established to be continuable in $\gamma > 0$ [7]. As we pathfollow T5 to nonzero γ the curve $E(\gamma)$ turns up to the branch of three-soliton complex $\psi_{(-+-)}$, see Fig.1b. Both branches on Fig.1b have been found to be unstable.

At the next step of numerical study we continued the strongly damped two-soliton complexes obtained in [1], to $\gamma < 0.5$. Since, in case $\gamma > 0.5$, solitons of (1) decay monotonically they cannot form bound states via the tail-overlap mechanism. Nevertheless, we obtained three two-soliton complexes with different distances (orbits) between constituents: $\psi_{1,(--)}$, $\psi_{2,(--)}$, $\psi_{3,(--)}$. They are shown on Fig. 2 for $\gamma = 0.49$, $h = 0.35$. As in the strong damping case, only $\psi_{2,(--)}$ has been found to be stable. Beside the stability analysis, these results were confirmed by means of direct numerical simulations, see Fig. 3. As we continue complexes $\psi_{2,(--)}$ in the direction $\gamma < 0.49$, the curve $E(\gamma)$ turns up into unstable branch of 4-soliton complex $\psi_{(+--+)}$. Continuing $\psi_{2,(--)}$ to $\gamma > 0.49$ we obtain unstable complex of two ψ_+-solitons.



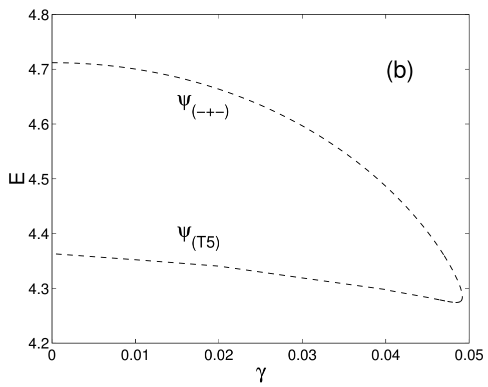


Figure 1. $h = 0.05$**: (a) Complex T5 at** $\gamma = 0$**; (b) Diagram** $E(\gamma)$ **obtained in continuation of T5 to** $\gamma > 0$

3. Double Sine-Gordon Equation: Effect of the 2nd Harmonic

The magnetic flux distributions in the case of a finite length overlap contact, satisfy the double sine-Gordon equation:

$$\varphi'' - \ddot{\varphi} - \alpha\dot{\varphi} = a_1 \sin\varphi + a_2 \sin 2\varphi - \gamma\,, \quad t > 0\,, \quad x \in (-l, l)\,, \quad \varphi'(\pm l, t) = h_e\,. \tag{3}$$

Here φ — magnetic flux distribution, h_e — external magnetic field, γ – external current, $\alpha \geqslant 0$ — dissipation coefficient, l — semilength of the junction, a_1 and a_2 —

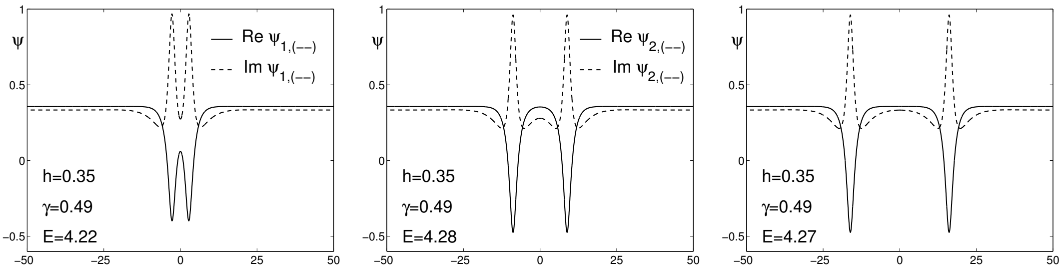


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Figure 2. Two-soliton complexes $\psi_{1,(--)}$, $\psi_{2,(--)}$, $\psi_{3,(--)}$ at $h = 0.35$; $\gamma = 0.49$

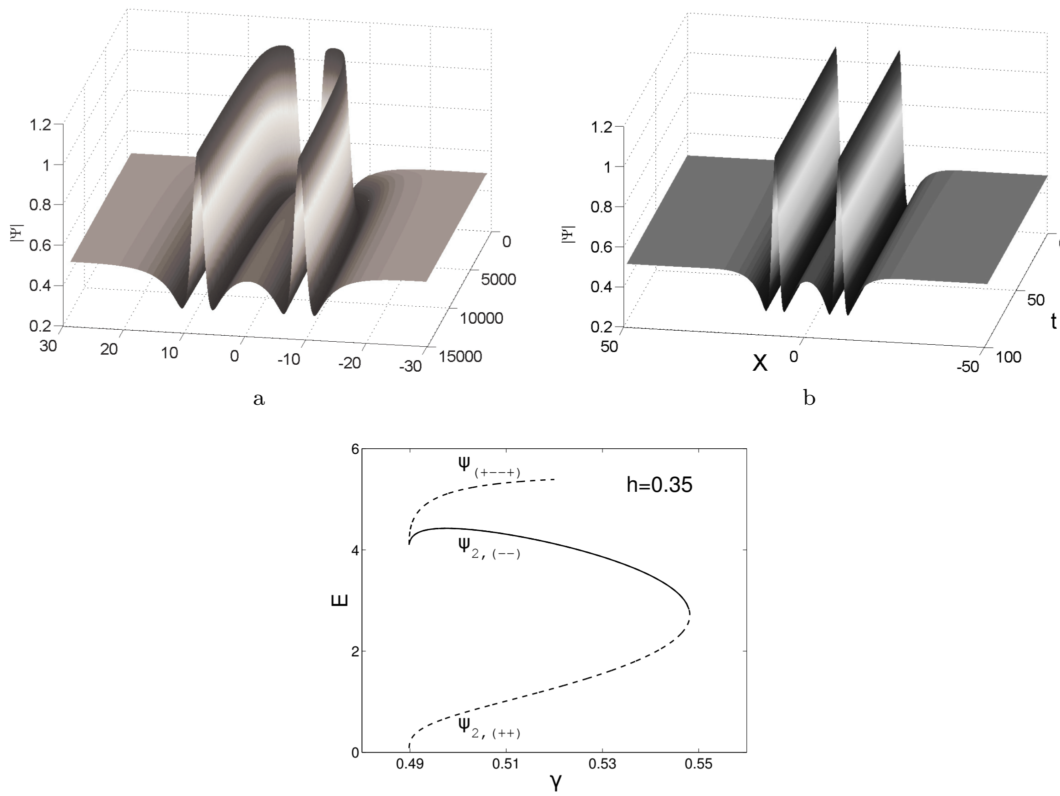

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Figure 3. Direct simulation of unstable complex $\psi_{1(--)}$ (left) and stable complex $\psi_{2,(--)}$ (centre) at $h = 0.35$; $\gamma = 0.49$. Right panel: Diagram $E(\gamma)$ obtained in continuation of $\psi_{2,(--)}$. Stable branch is shown by solid line

parameters of contribution of the first and second harmonics in the current-phase relation. Depending on the physical application, the sign of a_2 can be positive or negative. In this contribution, we consider only the case $a_2 < 0$.

The static magnetic flux distributions are described by the following boundary value problem:

$$-\varphi'' + a_1 \sin\varphi + a_2 \sin 2\varphi - \gamma = 0, \quad x \in (-l, l), \quad \varphi'(\pm l) = h_e. \tag{4}$$

Stability analysis can be reduced to numerical solution of the Sturm–Liouville problem [8]:

$$-\psi'' + q(x)\psi = \lambda\psi, \quad \psi'(\pm l) = 0, \quad \int\limits_{-l}^{+l} [\psi(x)]^2 \, \mathrm{d}x = 1, \qquad q(x) = a_1 \cos\varphi + 2a_2 \cos 2\varphi, \tag{5}$$

where the case $\lambda_0 > 0$ corresponds the stable solution φ. Eqs. (4), (5) are considered as the unified system with respect to unknown functions $\varphi(x)$, $\psi(x)$, and one of parameters l, a_1, a_2, h_e, γ. Putting $\lambda = 0$ one can obtain critical regimes of (3).

During the numerical continuation we calculate two quantities to characterize solutions:

– full magnetic flux $\Delta\varphi = \varphi(l) - \varphi(-l)$, and

– "number of fluxons" $N = [1/(2l\pi)] \int\limits_{-l}^{l} \varphi(x) \, \mathrm{d}x$.

Taking into account the second harmonic, i.e. introducing a nonzero a_2, changes the properties of the standard static magnetic flux distributions and gives rise to new (stable and unstable) static solutions [2, 9–12]. We plot the normalized rate of change of the magnetic flux $\Delta\varphi/2\pi$ versus the external magnetic field h_e in Fig.4 to demonstrate the connection between the coexisting stable and unstable solutions for $a_2 = -0.7$. Similarly to the case of $a_2 = 0$ [3] and a2=-0.5 [10], we plot two curves, the first of which connects Φ^1_{small} ($h_e = 0$) with multi-fluxon states φ^N with even "number on fluxons" N and the other one connects Φ^1_{large} ($h_e = 0$) with solutions φ^N with odd N. In addition we have found a short branch connecting Φ^1_{small} and M_{ac} which didn't exist in the case $a_2 = 0$ and $a_2 = -0.5$. This branch is seen in details on the right panel of Fig. 4. Stable and unstable solutions are plotted by solid and dashed lines, respectively. Light circles indicate turning points; solid circles show the points where stability changes.

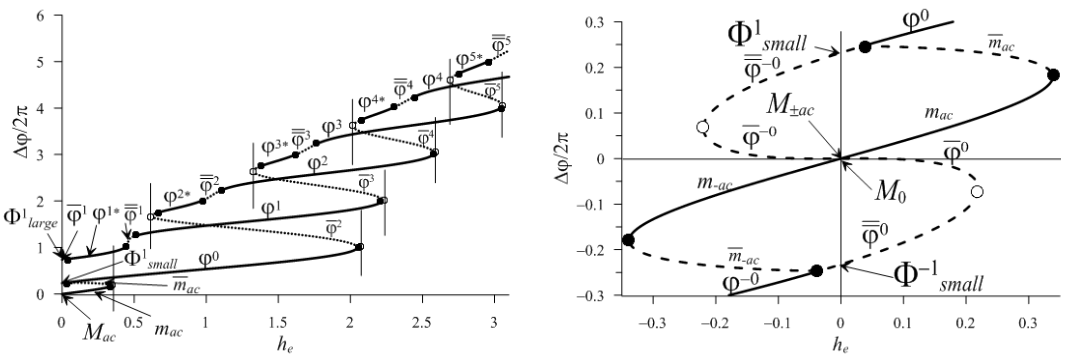

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Figure 4. Diagram $\Delta\varphi/2\pi(h_e)$ for $a_2 = -0.7$, $a_1 = 1$, $2l = 10$, $\gamma = 0$

Fig. 5 shows three coexisting solutions in case $h_e = 0$ (left panel). Only constant solution M_{ac} is stable here. When h_e is growing, "small" and "large" fluxons stabilize and complexity of coexisting solutions increases. Indeed, Fig. 5 (right panel) demonstrates four stable multi-fluxon solutions coexisting at $h_e = 1.5$ with two unstable solutions $\bar{\varphi}^2$ and $\bar{\varphi}^3$ (not plotted).

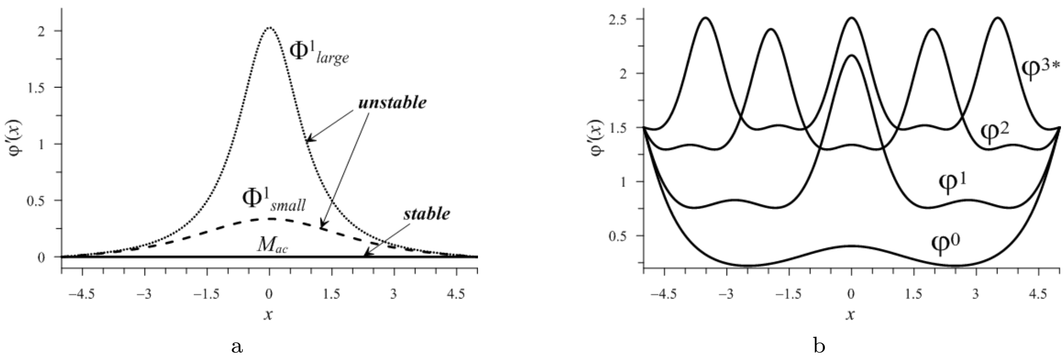

Figure 5. Coexisting stationary solutions of (4) for $a_2 = -0.7$ $a_1 = 1$, $2l = 10$, $\gamma = 0$ at $h_e = 0$ (left) and $h_e = 1.5$ (right)

4. Summary

Two dynamical systems (NLS and 2SG) have been investigated using the same numerical approach based on numerical continuation of stationary solutions. We show that in the weak damping case ($\gamma < 0.5$) (1) holds stable two-soliton complexes. We also show the 2nd harmonic increases the complexity of coexisting static distributions in the LJJs described by the 2SG equation.

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Комплексы локализованных структур в нелинейном уравнении Шрёдингера с диссипацией и прямой накачкой и в уравнении двойного синус-Гордона

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Проведено численное исследование комплексов локализованных структур в двух динамических системах, каждая из которых имеет множество физических приложений. Первая система описывается нелинейным уравнением Шредингера с внешней накачкой и диссипаций (NLS), вторая – уравнением двойного синус-Гордона (2SG). Численный анализ в обоих случаях основан на продолжении соответствующих стационарных решений по параметрам и численном решении линеаризованной задачи на собственные значения для анализа устойчивости и бифуркаций. Мультисолитонные комплексы NLS исследуются для случая слабой и нулевой диссипации. Для первой системы продемонстрировано существование устойчивых и неустойчивых мультисолитонных структур в случае малой диссипации. Численные результаты, полученные на основе вышеизложенного подхода, подтверждаются прямым численным решением исходного уравнения в частных производных. Для второй системы свойства мультифлюксонных решений 2SG исследованы в зависимости от параметра второй гармоники. Показано, что учет второй гармоники приводит к изменению свойств известных решений и появлению новых сосуществующих флюксонных состояний. Результаты обсуждаются применительно к модели длинных джозефсоновских контактов.

Ключевые слова: солитоны, флюксоны, ньютоновские итерации, численное продолжение, устойчивость.