UDC 512.64, 535.1, 535-4

Degenerate 4-Dimensional Matrices with Semi-Group Structure and Polarization Optics

E. M. Ovsiyuk*, V. M. Red'kov†

* *Mozyr State Pedagogical University, Belarus*
28 Studencheskaya Street 247760, Mozyr, Gomel region Belarus
† *Institute of Physics, NAS of Belarus*
220072 Minsk, Republic of Belarus, Nezavisimosty avenue, 68

In polarization optics, an important role play Mueller matrices — real four-dimensional matrices which describe the effect of action of optical elements on the polarization state of the light, described by 4-dimensional Stokes vectors. An important issue is to classify possible classes of the Mueller matrices. In particular, of special interest are degenerate Mueller matrices with vanishing determinants. With the use of a special technique of parameterizing arbitrary 4-dimensional matrices in Dirac basis, a classification of degenerate 4-dimensional real matrices of rank 1, 2, 3. is elaborated. To separate possible classes of degenerate matrices we impose linear restrictions on 16 parameters of 4×4 matrices which are compatible with the group multiplication law.

Key words and phrases: polarization of the light, degenerate Mueller matrices, classification.

In polarization optics, an important issue is to classify possible classes of the Mueller matrices — an extensive list of references on the subject is given in [1]. In particular, of special interest are degenerate Mueller matrices with vanishing determinants. There is known a special technique of parameterizing arbitrary 4-dimensional matrices with the use of four 4-dimensional vector (k, m, l, n) — see $[2, 3]$ and references therein. To separate possible simple classes of degenerate matrices of ranks 1, 2 and 3 we impose linear restrictions on (k, m, l, n), which are compatible with the group multiplication law. All the subsets of matrices obtained by this method, are either sub-groups or semigroups. To obtain singular matrices of rank 3, we specify 16 independent possibilities to get the 4-dimensional matrices with zero determinant.

In spinor basis, an arbitrary 4×4 matrix can be parameterized by four 4-dimensional vectors (k, m, l, n)

$$G = \begin{vmatrix} k_0 + \mathbf{k}\,\vec{\sigma} & n_0 + \mathbf{n}\,\vec{\sigma} \\ l_0 + \mathbf{l}\,\vec{\sigma} & m_0 + \mathbf{m}\,\vec{\sigma} \end{vmatrix} = \begin{vmatrix} K & N \\ L & M \end{vmatrix} . \tag{1}$$

The matrices G will be real, if the second components of parameters are imaginary

$$k_2^* = -k_2 \, , \quad m_2^* = -m_2 \, , \quad l_2^* = -l_2 \, , \quad n_2^* = -n_2 \, ,$$

and all remaining components are real. The law of multiplication for matrices given according to (1) in explicit form is

$$\begin{aligned} k''_0 &= k'_0\, k_0 + \mathbf{k}'\, \mathbf{k} + n'_0\, l_0 + \mathbf{n}'\, \mathbf{l} \, , \\ m''_0 &= m'_0\, m_0 + \mathbf{m}'\, \mathbf{m} + l'_0\, n_0 + \mathbf{l}'\, \mathbf{n} \, , \\ n''_0 &= k'_0\, n_0 + \mathbf{k}'\, \mathbf{n} + n'_0\, m_0 + \mathbf{n}'\, \mathbf{m} \, , \\ l''_0 &= l'_0\, k_0 + \mathbf{l}'\, \mathbf{k} + m'_0\, l_0 + \mathbf{m}'\, \mathbf{l} \, , \end{aligned}$$

Received 11^{th} June, 2012.

The work was supported by Belarusian Republican Foundation for Fundamental Research, Grant BRFFR-RA № F12RA–002.

$$\begin{aligned}
\mathbf{k}'' &= k_0'\,\mathbf{k} + \mathbf{k}'\,k_0 + i\,\mathbf{k}'\times\mathbf{k} + n_0'\,\mathbf{l} + \mathbf{n}'\,l_0 + i\,\mathbf{n}'\times\mathbf{l}\,,\\
\mathbf{m}'' &= m_0'\,\mathbf{m} + \mathbf{m}'\,m_0 + i\,\mathbf{m}'\times\mathbf{m} + l_0'\,\mathbf{n} + \mathbf{l}'\,n_0 + i\,\mathbf{l}'\times\mathbf{n}\,,\\
\mathbf{n}'' &= k_0'\,\mathbf{n} + \mathbf{k}'\,n_0 + i\,\mathbf{k}'\times\mathbf{n} + n_0'\,\mathbf{m} + \mathbf{n}'\,m_0 + i\,\mathbf{n}'\times\mathbf{m}\,,\\
\mathbf{l}'' &= l_0'\,\mathbf{k} + \mathbf{l}'\,k_0 + i\,\mathbf{l}'\times\mathbf{k} + m_0'\,\mathbf{l} + \mathbf{m}'\,l_0 + i\,\mathbf{m}'\times\mathbf{l}\,.
\end{aligned} \tag{2}$$

To separate possible sub-classes of matrices, we will impose linear restrictions on (k, m, l, n), which are compatible with the multiplication law (25).

First, consider the variants with one independent vector. **Variant I(k)**:

$$\begin{aligned}
\mathbf{n} &= A\,\mathbf{k}\,, \quad n_0 = \alpha\,k_0\,,\\
\mathbf{m} &= B\,\mathbf{k}\,, \quad m_0 = \beta\,k_0\,,\\
\mathbf{l} &= D\,\mathbf{k}\,, \quad l_0 = t\,k_0\,;
\end{aligned} \tag{3}$$

Formulas for multiplying parameters take the form

$$\begin{aligned}
k_0'' &= k_0'\,k_0 + \mathbf{k}'\,\mathbf{k} + \alpha t\,k_0'k_0 + AD\,\mathbf{k}'\mathbf{k}\,,\\
m_0'' &= \beta^2\,k_0'k_0 + B^2\,\mathbf{k}'\,\mathbf{k} + t\alpha\,k_0'\,k_0 + DA\,\mathbf{k}'\,\mathbf{k}\,,\\
n_0'' &= \alpha\,k_0'k_0 + A\,\mathbf{k}'\mathbf{k} + \alpha\beta\,k_0'\,k_0 + AB\,\mathbf{k}'\,\mathbf{k}\,,\\
l_0'' &= t\,k_0'\,k_0 + D\,\mathbf{k}'\,\mathbf{k} + \beta t\,k_0'k_0 + BD\,\mathbf{k}'\mathbf{k}\,,
\end{aligned} \tag{4}$$

$$\begin{aligned}
\mathbf{k}'' &= k_0'\,\mathbf{k} + \mathbf{k}'\,k_0 + i\,\mathbf{k}'\times\mathbf{k} + \alpha D\,k_0'\,\mathbf{k} + At\,\mathbf{k}'k_0 + iAD\,\mathbf{k}'\times\mathbf{k}\,,\\
\mathbf{m}'' &= \beta B\,k_0'\mathbf{k} + B\beta\,\mathbf{k}'\,k_0 + iB^2\,\mathbf{k}'\times\mathbf{k} + tA\,k_0'\mathbf{k} + D\alpha\,\mathbf{k}'\,k_0 + iDA\,\mathbf{k}'\times\mathbf{k}\,,\\
\mathbf{n}'' &= A\,k_0'\,\mathbf{k} + \alpha\,\mathbf{k}'\,k_0 + iA\,\mathbf{k}'\times\mathbf{k} + \alpha B\,k_0'\mathbf{k} + A\beta\,\mathbf{k}'\,k_0 + iAB\,\mathbf{k}'\times\mathbf{k}\,,\\
\mathbf{l}'' &= t\,k_0'\mathbf{k} + D\,\mathbf{k}'k_0 + iD\,\mathbf{k}'\times\mathbf{k} + \beta D\,k_0'\,\mathbf{k} + Bt\,\mathbf{k}'\,k_0 + iBD\,\mathbf{k}'\times\mathbf{k}\,.
\end{aligned} \tag{5}$$

Twice primed parameters must obey the same restrictions (3) from that it follows $\underline{n_0'' = \alpha\,k_0''\,,}$

$$\alpha(1+\beta)\,k_0'k_0 + A(1+B)\,\mathbf{k}'\mathbf{k} = \alpha(1+\alpha t)\,k_0'\,k_0 + \alpha(1+AD)\,\mathbf{k}'\,\mathbf{k}\,,$$

$\underline{m_0'' = \beta\,k_0''\,,}$

$$(\beta^2 + t\alpha)\,k_0'k_0 + (B^2 + DA)\,\mathbf{k}'\,\mathbf{k} = \beta(1+\alpha t)\,k_0'\,k_0 + \beta(1+AD)\,\mathbf{k}'\,\mathbf{k}\,,$$

$\underline{l_0'' = t\,k_0''\,,}$

$$t(1+\beta)\,k_0'\,k_0 + D(1+B)\,\mathbf{k}'\,\mathbf{k} = t\,(1+\alpha t)\,k_0'\,k_0 + t(1+AD)\,\mathbf{k}'\,\mathbf{k}\,.$$

$\underline{\mathbf{n}'' = A\,\mathbf{k}''\,,}$

$$\begin{aligned}
&(A+\alpha B)\,k_0'\,\mathbf{k} + (\alpha + A\beta)\,\mathbf{k}'\,k_0 + iA(1+B)\,\mathbf{k}'\times\mathbf{k} =\\
&\qquad = A(1+\alpha D)\,k_0'\,\mathbf{k} + A(1+At)\,\mathbf{k}'\,k_0 + iA(1+AD)\,\mathbf{k}'\times\mathbf{k}\,,
\end{aligned}$$

$\underline{\mathbf{m}'' = B\,\mathbf{k}''\,,}$

$$\begin{aligned}
&(\beta B + tA)\,k_0'\mathbf{k} + (B\beta + D\alpha)\,\mathbf{k}'\,k_0 + i(B^2 + AD)\,\mathbf{k}'\times\mathbf{k} =\\
&\qquad = B(1+\alpha D)k_0'\,\mathbf{k} + B(1+At)\,\mathbf{k}'\,k_0 + iB(1+AD)\,\mathbf{k}'\times\mathbf{k}\,,
\end{aligned}$$

$\underline{\mathbf{l}'' = D\,\mathbf{k}''\,,}$

$$(t+\beta D)\ k_0'\mathbf{k} + (D+Bt)\ \mathbf{k}'k_0 + iD(1+B)\ \mathbf{k}'\times\mathbf{k} =$$
$$= D(1+\alpha D)\ k_0'\ \mathbf{k} + D(1+At)\ \mathbf{k}'\ k_0 + iD(1+AD)\ \mathbf{k}'\times\mathbf{k}\ .$$

Further we get the system of algebraic equations

$$\alpha(1+\beta) = \alpha(1+\alpha t)\ , \qquad A(1+B) = \alpha(1+AD)\ ,$$
$$(\beta^2 + t\alpha) = \beta(1+\alpha t)\ , \qquad (B^2 + DA) = \beta(1+AD)\ ,$$
$$t(1+\beta) = t(1+\alpha t)\ , \qquad D(1+B) = t(1+AD)\ ,$$

$$(A+\alpha B) = A(1+\alpha D)\ , \qquad (\alpha + A\beta) = A(1+At)\ ,$$
$$(\beta B + tA) = B(1+\alpha D)\ , \qquad (B\beta + D\alpha) = B(1+At)\ ,$$
$$(t+\beta D) = D(1+\alpha D)\ , \qquad (D+Bt) = D(1+At)\ ,$$
$$A(1+B) = A(1+AD)\ ,$$
$$(B^2 + AD) = B(1+AD)\ ,$$
$$D(1+B) = D(1+AD)\ .$$

We are to find all solutions of Eqs. (25); each of them will represent a sub-group or semi-group.

First, we note a trivial case, solution $(K-1)$:

$$A=\alpha=0\ , \quad B=\beta=0\ , \quad D=t=0\ , \quad G = \left|\begin{array}{cc} K & 0 \\ 0 & 0 \end{array}\right|\ ; \tag{6}$$

all such matrices are degenerate; their determinant equals to zero; The rank of the matrices equals to 2; if $\det K = 0$ then the rank equals to 1.

Let us consider possibilities at two vanishing blocks — there are three different cases. First, let it be

$$A=\alpha=0\ , \quad D=t=0\ ;$$

Eqs. (25) give

$$\beta^2 = \beta\ , \quad B^2 = \beta\ , \quad \beta B = B\ , \quad B^2 = B\ ; \tag{7}$$

from whence we get one new solution (in addition to yet found trivial one (6) at $B=\beta=0$)

solution $(K-2)$,

$$A=\alpha=0\ , \quad D=t=0\ , \quad D=\beta=+1\ , \quad G = \left|\begin{array}{cc} k_0+\mathbf{k}\vec{\sigma} & 0 \\ 0 & k_0+\mathbf{k}\vec{\sigma} \end{array}\right|; \tag{8}$$

which determines a set with group structure.

Now, let us suppose

$$A=\alpha=0\ , \quad B=\beta=0\ ; \tag{9}$$

then Eqs. (25) lead

$$\begin{array}{ccc} & 0=0\ , & A=0\ , \\ & 0=0\ , & 0=0\ , \\ & t=t\ , & D=t\ , \\ 0=0\ , & 0=0\ , & 0=0\ , \\ 0=0\ , & 0=0\ , & 0=0\ , \\ t=D\ , & D=D\ , & D=D\ ; \end{array}$$

so we obtain one new $\underline{\text{solution } (K-3)}$:

$$A = \alpha = 0\,, \quad B = \beta = 0\,, \quad D = t\,, \quad G = \begin{vmatrix} K & 0 \\ DK & 0 \end{vmatrix}, \tag{10}$$

with simple multiplication law

$$G'G = \begin{vmatrix} K' & 0 \\ DK' & 0 \end{vmatrix} \begin{vmatrix} K & 0 \\ DK & 0 \end{vmatrix} = \begin{vmatrix} K'K & 0 \\ DK'K & 0 \end{vmatrix}.$$

It is a set of degenerate matrices with semi-group structure (with a free parameter D).

Now let us suppose

$$B = \beta = 0\,, \quad D = t = 0\,;$$

the system (25) gives

$$\begin{array}{ccc} & \alpha = \alpha\,, & A = \alpha\,, \\ & 0 = 0\,, & 0 = 0\,, \\ & 0 = 0\,, & 0 = 0\,, \\ A = A\,, & \alpha = A\,, & A = A\,, \\ 0 = 0\,, & 0 = 0\,, & 0 = 0\,, \\ 0 = 0\,, & 0 = 0\,, & 0 = 0\,, \end{array}$$

so we arrive at a new $\underline{\text{solution } (K-4)}$,

$$A = \alpha\,, \quad B = \beta = 0\,, \quad D = t = 0\,, \quad G = \begin{vmatrix} K & AK \\ 0 & 0 \end{vmatrix}, \tag{11}$$

with simple multiplication law

$$G'G = \begin{vmatrix} K' & AK' \\ 0 & 0 \end{vmatrix} \begin{vmatrix} K & AK \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} K'K & AK'K \\ 0 & 0 \end{vmatrix}.$$

It is a set of degenerate matrices with semi-group structure (with a free parameter A).

Now, we are do consider variants wit one vanishing block. First possibility is

$$A = \alpha = 0\,;$$

the system (25) leads to

$$\begin{array}{rrr} & 0 = 0\,, & 0 = 0\,, \\ & \beta^2 = \beta\,, & B^2 = \beta\,, \\ & t(1+\beta) = t\,, & D(1+B) = t\,, \\ 0 = 0\,, & 0 = 0\,, & 0 = 0\,, \\ \beta B = B\,, & B\beta = B\,, & B^2 = B\,, \\ (t + \beta D) = D\,, & (D + Bt) = D\,, & D(1+B) = D\,; \end{array}$$

it provides us with two yet known solutions

$$\underline{(K-3)}, \quad A = \alpha = 0\,, \quad B = \beta = 0\,, \quad t = D\,, \quad G = \begin{vmatrix} K & 0 \\ DK & 0 \end{vmatrix},$$

$$\underline{(K-2)}, \quad A = \alpha = 0\,, \quad B = \beta = +1\,, \quad D = t = 0\,, \quad G = \left| \begin{array}{cc} k_0 + \mathbf{k}\vec{\sigma} & 0 \\ 0 & k_0 + \mathbf{k}\vec{\sigma} \end{array} \right| .$$

The second possibility with one vanishing block is

$$D = t = 0\,;$$

Eqs. (25) give

$$\begin{array}{rrr} & \alpha\beta = 0\,, & A(1+B) = \alpha\,, \\ & \beta^2 = \beta\,, & B^2 = \beta\,, \\ & 0 = 0\,, & 0 = 0\,, \\ (A + \alpha B) = A\,, & (\alpha + A\beta) = A\,, & AB = 0\,, \\ \beta B = B\,, & B\beta = B\,, & B^2 = B\,, \\ 0 = 0\,, & 0 = 0\,, & 0 = 0\,. \end{array}$$

which leads us to yet known solutions:

$$\underline{(K-2)}, \quad A = \alpha = 0\,, \quad B = +1\,, \quad \beta = +1\,, \quad D = t = 0\,, \quad G = \left| \begin{array}{cc} k_0 + \mathbf{k}\vec{\sigma} & 0 \\ 0 & k_0 + \mathbf{k}\vec{\sigma} \end{array} \right| .$$

$$\underline{(K-1)}, A = \alpha\,, \quad B = \beta = 0\,, \quad dD = t = 0\,, \quad G = \left| \begin{array}{cc} K & 0 \\ 0 & 0 \end{array} \right| .$$

Consider the third possibility wit hone vanishing block

$$B = \beta = 0\,;$$

Eqs. (25) give

$$\begin{array}{rrr} & 0 = \alpha^2 t\,, & A = \alpha(1 + AD)\,, \\ & t\alpha = 0\,, & DA = 0\,, \\ & 0 = \alpha t^2\,, & D = t(1 + AD)\,, \\ 0 = A\alpha D\,, & \alpha = A(1 + At)\,, & 0 = A^2 D\,, \\ tA = 0\,, & D\alpha = 0\,, & AD = 0\,, \\ t = D(1 + \alpha D)\,, & 0 = DAt\,, & 0 = AD^2\,. \end{array}$$

From whence we arrive at two known solutions:

$$\underline{(K-4)}, \quad A = \alpha\,, \quad B = \beta = 0\,, \quad D = t = 0\,, \quad G = \left| \begin{array}{cc} K & AK \\ 0 & 0 \end{array} \right| ;$$

$$\underline{(K-3)}, \quad A = \alpha = 0\,, \quad B = \beta = 0\,, \quad D = t\,, \quad G = \left| \begin{array}{cc} K & 0 \\ DK & 0 \end{array} \right| .$$

It remains to consider equations (25) if all blocks are non-vanishing.

$$A, \quad \alpha \neq 0\,, \qquad B, \quad \beta \neq 0\,, \qquad D, \quad t \neq 0\,;$$

Eqs. (25) lead to

$$\begin{array}{ccc}
 & \beta = +\alpha t \ , & A(1+B) = \alpha(1+AD) \ , \\
 & (\beta^2 + t\alpha) = \beta(1+\alpha t) \ , & (B^2 + DA) = \beta(1+AD) \ , \\
 & \beta = +\alpha t \ , & D(1+B) = t(1+AD) \ , \\
B = AD \ , & (\alpha + A\beta) = A(1+At) \ , & B = AD \ , \\
(\beta B + tA) = B(1+\alpha D) \ , & (B\beta + D\alpha) = B(1+At) \ , & (B^2 + AD) = B(1+AD) \ , \\
(t + \beta D) = D(1+\alpha D) \ , & B = AD \ , & B = AD \ .
\end{array}$$

Excluding the variables B and β, $B = AD, \beta = \alpha t$; remaining independent equations are

$$\begin{gathered}
(A-\alpha)(1+AD) = 0 \ , \\
(AD - \alpha t)(1+AD) = 0 \ , \\
(D-t)(1+AD) = 0 \ , \\
(A-\alpha)(1+At) = 0 \ , \\
(D-t)(1+\alpha D) = 0 \ .
\end{gathered} \tag{12}$$

First, let us consider the case $A = \alpha$; Eqs. (12) take the form

$$A = \alpha \ , \quad (D-t)(1+AD) = 0 \ ;$$

from whence we get two new solutions.

<u>Solution $(K-5)$</u>,

$$A = \alpha \ , \quad B = \beta = AD \ , \quad D = t \ , \quad G = \left| \begin{array}{cc} K & AK \\ DK & ADK \end{array} \right| ;$$

it is a set of degenerate matrices (with two additional free parameters, A and D) of the rank 2 with simple multiplication law

$$\left| \begin{array}{cc} K' & AK' \\ DK' & ADK' \end{array} \right| \left| \begin{array}{cc} K & AK \\ DK & ADK \end{array} \right| = \left| \begin{array}{cc} (K'K + ADK'K) & A(K'K + ADK'K) \\ D(K'K + ADK'K) & AD(K'K + ADK'K) \end{array} \right| ;$$

<u>Solution $(K-6)$</u>,

$$\begin{gathered}
A = \alpha \ , \quad B = -1, \quad \beta = -At, \quad D = -A^{-1} \ , \\
G = \left| \begin{array}{cc} k_0 + \vec{k}\vec{\sigma} & Ak_0 + A\vec{k}\vec{\sigma} \\ tk_0 - A^{-1}\vec{k}\vec{\sigma} & Atk_0 - \vec{k}\vec{\sigma} \end{array} \right| .
\end{gathered} \tag{13}$$

it is a set of degenerate matrices of rank 2 (with two free parameters A and t).

Now, turning back to (12) let us consider the case

$$A \neq \alpha \ , \quad 1 + AD = 0 \ ; \tag{14}$$

the system (12) leads to a single solution

$$\underline{(K-7)}, \qquad A, \alpha, \quad B=-1, \quad \beta=-\frac{\alpha}{A}\,, \quad D=t=-\frac{1}{A}$$
$$G=\begin{vmatrix} k_0+\vec{k}\vec{\sigma} & \alpha k_0+A\vec{k}\vec{\sigma} \\ -A^{-1}(k_0+\vec{k}\vec{\sigma}) & -A^{-1}\alpha k_0-\vec{k}\vec{\sigma} \end{vmatrix}; \qquad (15)$$

they all are degenerate matrices with rank 2. Let us verify the multiplication law:

$$G'G=\begin{vmatrix} k'_0+\vec{k}'\vec{\sigma} & \alpha k'_0+A\vec{k}'\vec{\sigma} \\ -A^{-1}(k'_0+\vec{k}'\vec{\sigma}) & -A^{-1}\alpha k'_0-\vec{k}'\vec{\sigma} \end{vmatrix}\begin{vmatrix} k_0+\vec{k}\vec{\sigma} & \alpha k_0+A\vec{k}\vec{\sigma} \\ -A^{-1}(k_0+\vec{k}\vec{\sigma}) & -A^{-1}\alpha k_0-\vec{k}\vec{\sigma} \end{vmatrix};$$

result is better to present by blocks

$$(11)=\Big(1-\frac{\alpha}{A}\Big)\,k'_0k_0+\Big(1-\frac{\alpha}{A}\Big)k'_0\vec{k}\,\vec{\sigma}\,,$$
$$(12)=\alpha\,\Big(1-\frac{\alpha}{A}\Big)\,k'_0k_0+A\,\Big(1-\frac{\alpha}{A}\Big)k'_0\vec{k}\,\vec{\sigma}\,,$$
$$(21)=-A^{-1}\,\Big(1-\frac{\alpha}{A}\Big)\,k'_0k_0-A^{-1}\,\Big(1-\frac{\alpha}{A}\Big)k'_0\vec{k}\,\vec{\sigma}\,,$$
$$(22)=-A^{-1}\alpha\,\Big(1-\frac{\alpha}{A}\Big)\,k'_0k_0-\Big(1-\frac{\alpha}{A}\Big)k'_0\vec{k}\,\vec{\sigma}\,,$$

the multiplication law in this semi-group looks as follows

$$G''=G'G\,, \quad k''_0=\Big(1-\frac{\alpha}{A}\Big)\,k'_0k_0\,, \quad \vec{k}''=\Big(1-\frac{\alpha}{A}\Big)k'_0\vec{k}\,.$$

Let us collect results together (there exist only 7 solutions):

$$G=\begin{vmatrix} K & 0 \\ 0 & 0 \end{vmatrix}, \qquad G=\begin{vmatrix} k_0+\mathbf{k}\vec{\sigma} & 0 \\ 0 & k_0+\mathbf{k}\vec{\sigma} \end{vmatrix},$$
$$G=\begin{vmatrix} K & 0 \\ DK & 0 \end{vmatrix}, \qquad G=\begin{vmatrix} K & AK \\ 0 & 0 \end{vmatrix},$$
$$G=\begin{vmatrix} K & AK \\ DK & ADK \end{vmatrix}, \qquad G=\begin{vmatrix} k_0+\vec{k}\vec{\sigma} & Ak_0+A\vec{k}\vec{\sigma} \\ tk_0-A^{-1}\vec{k}\vec{\sigma} & Atk_0-\vec{k}\vec{\sigma} \end{vmatrix}, \qquad (16)$$
$$G=\begin{vmatrix} k_0+\vec{k}\vec{\sigma} & \alpha k_0+A\vec{k}\vec{\sigma} \\ -A^{-1}(k_0+\vec{k}\vec{\sigma}) & -A^{-1}\alpha k_0-\vec{k}\vec{\sigma} \end{vmatrix}.$$

Here there are only 7 types of solutions, among 7 types of solutions, 6 cases lead to the structure of semigroup (matrices with rank 2).

For all other variants we will write down only final results.

Variant I(m)

$$\mathbf{n}=A\,\mathbf{m}\,, \quad n_0=\alpha\,m_0\,,$$
$$\mathbf{k}=B\,\mathbf{m}\,, \quad k_0=\beta\,m_0\,, \qquad (17)$$
$$\mathbf{l}=D\,\mathbf{m}\,, \quad l_0=t\,m_0\,;$$

there exist only 7 solutions:

$$G = \begin{vmatrix} 0 & 0 \\ 0 & M \end{vmatrix}, \qquad G = \begin{vmatrix} M & 0 \\ 0 & M \end{vmatrix},$$
$$G = \begin{vmatrix} 0 & 0 \\ DM & M \end{vmatrix}, \qquad G = \begin{vmatrix} 0 & AM \\ 0 & M \end{vmatrix},$$
$$G = \begin{vmatrix} (Atm_0 - \vec{m}\vec{\sigma}) & (Am_0 + A\vec{m}\vec{\sigma}) \\ (tm_0 - A^{-1}\vec{m}\vec{\sigma}) & (m_0 + \vec{m}\vec{\sigma}) \end{vmatrix}, \tag{18}$$
$$G = \begin{vmatrix} (-\alpha A^{-1}tm_0 - \vec{m}\vec{\sigma}) & (\alpha m_0 + A\vec{m}\vec{\sigma}) \\ (A^{-1}m_0 + A^{-1}\vec{m}\vec{\sigma}) & (m_0 + \vec{m}\vec{\sigma}) \end{vmatrix},$$
$$G = \begin{vmatrix} ADM & AM \\ DM & M \end{vmatrix}.$$

Here again 6 cases describe semigroups of the rank 2.

Variant I(n)

$$\begin{aligned} \mathbf{k} &= A\,\mathbf{n}\,, \quad & k_0 &= \alpha\, n_0\,, \\ \mathbf{m} &= B\,\mathbf{n}\,, \quad & m_0 &= \beta\, n_0\,, \\ \mathbf{l} &= D\,\mathbf{n}\,, \quad & l_0 &= t\, n_0\,; \end{aligned} \tag{19}$$

there exist only 4 solutions:

$$G = \begin{vmatrix} AN & N \\ 0 & 0 \end{vmatrix}, \qquad G = \begin{vmatrix} AN & N \\ A^2N & AN \end{vmatrix},$$
$$G = \begin{vmatrix} \alpha n_0 + A\mathbf{n}\vec{\sigma} & n_0 + \mathbf{n}\vec{\sigma} \\ -\alpha A n_0 - A^2\mathbf{n}\vec{\sigma} & -An_0 - A\mathbf{n}\vec{\sigma} \end{vmatrix}, \tag{20}$$
$$G = \begin{vmatrix} An_0 + A\mathbf{n}\vec{\sigma} & n_0 + \mathbf{n}\vec{\sigma} \\ \beta A n_0 - A^2\mathbf{n}\vec{\sigma} & \beta n_0 - A\mathbf{n}\vec{\sigma} \end{vmatrix}.$$

All four solutions describe semigroup of the rank 2.

Variant I(l):

$$\begin{aligned} \mathbf{k} &= A\,\mathbf{l}\,, \quad & k_0 &= \alpha\, l_0\,, \\ \mathbf{m} &= B\,\mathbf{l}\,, \quad & m_0 &= \beta\, l_0\,, \\ \mathbf{n} &= D\,\mathbf{l}\,, \quad & n_0 &= t\, l_0\,; \end{aligned} \tag{21}$$

there exist only 4 solutions:

$$G = \begin{vmatrix} AL & 0 \\ L & 0 \end{vmatrix}, \quad G = \begin{vmatrix} AL & A^2L \\ L & AL \end{vmatrix},$$
$$G = \begin{vmatrix} \alpha l_0 + A\mathbf{l}\vec{\sigma} & -\alpha A l_0 - A^2\mathbf{l} \\ l_0 + \mathbf{l}\vec{\sigma} & -Al_0 - A\mathbf{l}\vec{\sigma} \end{vmatrix}, \quad G = \begin{vmatrix} Al_0 + A\mathbf{l}\vec{\sigma} & \beta A l_0 - A^2\mathbf{l}\vec{\sigma} \\ l_0 + \mathbf{l}\vec{\sigma} & \beta l_0 - A\mathbf{l}\vec{\sigma} \end{vmatrix}. \tag{22}$$

All four solutions describe semigroup of rank 2.

We now consider the cases of two independent vectors.

Variant II(k, m)

$$\begin{aligned} \mathbf{n} = A\mathbf{k} + B\mathbf{m}\,, \quad n_0 = \alpha k_0 + \beta m_0\,, \\ \mathbf{l} = C\mathbf{k} + D\mathbf{m}\,, \quad l_0 = s k_0 + t m_0\,; \end{aligned} \tag{23}$$

there exist only 5 solutions:

$$\begin{gathered} G = \begin{vmatrix} K & 0 \\ 0 & M \end{vmatrix}, \quad G = \begin{vmatrix} K & 0 \\ D(M-K) & M \end{vmatrix}, \quad G = \begin{vmatrix} K & BM \\ B^{-1}K & M \end{vmatrix}, \\ G = \begin{vmatrix} K & A(K-M) \\ 0 & M \end{vmatrix}, \qquad G = \begin{vmatrix} K & A(K-M) \\ C(K-M) & M \end{vmatrix}. \end{gathered} \tag{24}$$

All solutions except for the $(KM - 1)$ describe semigroups of rank 2.

Variant II(l, n)

$$\begin{aligned} \mathbf{k} = (A\mathbf{l} + B\mathbf{n})\,, \quad k_0 = (\alpha l_0 + \beta n_0)\,, \\ \mathbf{m} = (D\mathbf{l} + C\mathbf{n})\,, \quad m_0 = (t l_0 + s n_0)\,; \end{aligned} \tag{25}$$

there exist only 2 solutions:

$$G = \begin{vmatrix} AL & N \\ L & A^{-1}N \end{vmatrix}, \quad G = \begin{vmatrix} BN & N \\ L & B^{-1}L \end{vmatrix}. \tag{26}$$

The two solutions describe semigroups of rank 2.

Variant II(k, n)

$$\begin{aligned} \mathbf{l} = (A\mathbf{k} + B\mathbf{n})\,, \quad l_0 = (\alpha k_0 + \beta n_0)\,, \\ \mathbf{m} = (D\mathbf{n} + C\mathbf{k})\,, \quad m_0 = (t n_0 + s k_0)\,; \end{aligned} \tag{27}$$

solutions:

$$G = \begin{vmatrix} K & N \\ AK & AN \end{vmatrix}, \quad G = \begin{vmatrix} K & N \\ 0 & K \end{vmatrix}. \tag{28}$$

The first Solution describes a semigroup of rank 2, the second describe a group.

Variant II(m, l)

$$\begin{aligned} \mathbf{n} = A\mathbf{m} + B\mathbf{l}\,, \quad l_0 = \alpha m_0 + \beta l_0\,, \\ \mathbf{k} = D\mathbf{l} + C\mathbf{m}\,, \quad k_0 = t l_0 + s m_0\,; \end{aligned} \tag{29}$$

there exist only 2 solutions:

$$G = \begin{vmatrix} AL & AM \\ L & M \end{vmatrix}, \quad G = \begin{vmatrix} M & 0 \\ L & M \end{vmatrix}. \tag{30}$$

The first solution describes the semigroup of rank 2, the second — the group.

Variant II(k, l)

$$\begin{aligned} \mathbf{n} = A\mathbf{k} + B\mathbf{l}\,, \quad n_0 = \alpha k_0 + \beta l_0\,, \\ \mathbf{m} = D\mathbf{l} + C\mathbf{k}\,, \quad m_0 = t l_0 + s k_0\,; \end{aligned} \tag{31}$$

there exist only 2 solutions:

$$G = \begin{vmatrix} K & AK \\ L & AL \end{vmatrix}, \quad G = \begin{vmatrix} K & L \\ L & K+L \end{vmatrix}. \tag{32}$$

Variant II(n, m)

$$\begin{aligned} \mathbf{l} &= A\mathbf{m} + B\mathbf{n} \ , \quad n_0 = \alpha m_0 + \beta n_0 \ , \\ \mathbf{k} &= D\mathbf{n} + C\mathbf{m} \ , \quad k_0 = t n_0 + s m_0 \ ; \end{aligned} \tag{33}$$

there exist only 2 solutions:

$$G = \begin{vmatrix} AN & N \\ AM & M \end{vmatrix} , \quad G = \begin{vmatrix} M+N & N \\ N & M \end{vmatrix} . \tag{34}$$

Now consider the case 3 independent vectors.

Variant I(k, m, n):

$$\mathbf{l} = A\mathbf{k} + B\mathbf{m} + C\mathbf{n} \ , \quad l_0 = \alpha k_0 + \beta m_0 + s n_0 \ ; \tag{35}$$

there exist only 2 solutions:

$$G = \begin{vmatrix} K & N \\ 0 & M \end{vmatrix} , \quad G = \begin{vmatrix} K & N \\ -K+M+N & M \end{vmatrix} . \tag{36}$$

The first solution describes the group, the second — the semigroup of rank 2.

There are similar solutions for

variant I(k, m, l):

$$G = \begin{vmatrix} K & 0 \\ L & M \end{vmatrix} , \quad G = \begin{vmatrix} K & -M+K+L \\ L & M \end{vmatrix} . \tag{37}$$

Variant I(n, l, k)

$$\mathbf{m} = A\mathbf{n} + B\mathbf{l} + C\mathbf{k} \ , \quad m_0 = \alpha n_0 + \beta l_0 + s k_0 \ ; \tag{38}$$

there exists only 1 solution:

$$G = \begin{vmatrix} K & N \\ L & (K + AN - A^{-1}L) \end{vmatrix} \tag{39}$$

it describes a semigroup of rank 2.

There is a similar solution for **variant I(n, l, m)**

$$G = \begin{vmatrix} (M + AL - A^{-1}N) & N \\ L & M \end{vmatrix} ; \tag{40}$$

it describes a semigroup of rank 2.

In all cases above, from the semigroups of the rank 2 one can easily to obtain semigroups of the rank 1, for this it suffices to add a requirement that the determinant of the basic 2×2-matrices be equal to zero.

Let us consider singular Mueller matrices of the rank 3. Given the explicit form of the matrix G, it is easy to understand that there are 16 simple ways to get the semigroups of rank 3. It is enough to have vanishing any i-line and any j-column in the original 4-dimensional matrix. Compatibility of the law of multiplication with this constrain is obvious. All 16 possibilities are listed below.

Variant (00)

$$G = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 2k_0 & 2n_1 & 2n_0 \\ 0 & 2l_1 & m_0 + m_3 & m_1 - im_2 \\ 0 & 2l_0 & m_1 + im_2 & m_0 - m_3 \end{array} \right| ,$$

$$k_1 = 0, \quad k_2 = 0, \quad k_0 = -k_3 \ ,$$

$$n_0 = -n_3, \quad l_0 = -l_3, \quad +in_2 = n_1, \quad -il_2 = l_1 \ .$$

Variant (01)

$$G = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 2k_1 & 0 & 2n_1 & 2n_0 \\ 2l_0 & 0 & m_0 + m_3 & m_1 - im_2 \\ 2l_1 & 0 & m_1 + im_2 & m_0 - m_3 \end{array} \right| ,$$

$$k_0 = 0, \quad k_3 = 0, \quad k_1 = ik_2 \ ,$$

$$l_1 = il_2, \quad l_0 = l_3, \quad n_0 = -n_3, \quad n_1 = in_2 \ .$$

Variant (02)

$$G = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 2k_1 & 2k_0 & 0 & 2n_0 \\ l_0 + l_3 & l_1 - il_2 & 0 & 2m_1 \\ l_1 + il_2 & l_0 - l_3 & 0 & 2m_0 \end{array} \right| ,$$

$$n_1 = 0, \quad n_2 = 0, \quad n_0 = -n_3 \ ,$$

$$m_0 = -m_3, \quad m_1 = -im_2, \quad k_0 = -k_3, \quad k_1 = ik_2 \ .$$

Variant (03)

$$G = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 2k_1 & 2k_0 & 2n_1 & 0 \\ l_0 + l_3 & l_1 - il_2 & 2m_0 & 0 \\ l_1 + il_2 & l_0 - l_3 & 2m_1 & 0 \end{array} \right| ,$$

$$n_0 = 0, \quad n_3 = 0, \quad n_1 = in_2 \ ,$$

$$m_0 = m_3, \quad m_1 = im_2, \quad k_0 = -k_3, \quad k_1 = ik_2 \ .$$

Variant (10)

$$G = \left| \begin{array}{cccc} 0 & 2k_1 & 2n_0 & 2n_1 \\ 0 & 0 & 0 & 0 \\ 0 & 2l_1 & m_0 + m_3 & m_1 - im_2 \\ 0 & 2l_0 & m_1 + im_2 & m_0 - m_3 \end{array} \right| ,$$

$$k_0 = 0, \quad k_3 = 0, \quad k_1 = -ik_2 \ ,$$

$$l_0 = -l_3, \quad l_1 = -il_2, \quad n_1 = -in_2, \quad n_0 = n_3 \ .$$

Variant (11)

$$G = \begin{vmatrix} 2k_0 & 0 & 2n_0 & 2n_1 \\ 0 & 0 & 0 & 0 \\ 2l_0 & 0 & m_0 + m_3 & m_1 - im_2 \\ 2l_1 & 0 & m_1 + im_2 & m_0 - m_3 \end{vmatrix},$$
$$k_1 = 0, \quad k_2 = 0, \quad k_0 = k_3 \ ,$$
$$l_0 = l_3, \quad l_1 = il_2, \quad n_1 = -in_2, \quad n_0 = n_3 \ .$$

Variant (12)

$$G = \begin{vmatrix} 2k_0 & 2k_1 & 0 & 2n_1 \\ 0 & 0 & 0 & 0 \\ l_0 + l_3 & l_1 - il_2 & 0 & 2m_1 \\ l_1 + il_2 & l_0 - l_3 & 0 & 2m_0 \end{vmatrix},$$
$$n_0 = 0, \quad n_3 = 0, \quad n_1 = -in_2 \ ,$$
$$m_0 = -m_3, \quad m_1 = -im_2, \quad k_1 = -ik_2, \quad k_0 = k_3 \ .$$

Variant (13)

$$G = \begin{vmatrix} 2k_0 & 2k_1 & 2n_0 & 0 \\ 0 & 0 & 0 & 0 \\ l_0 + l_3 & l_1 - il_2 & 2m_0 & 0 \\ l_1 + il_2 & l_0 - l_3 & 2m_1 & 0 \end{vmatrix},$$
$$n_1 = 0, \quad n_2 = 0, \quad n_0 = n_3 \ ,$$
$$m_0 = m_3, \quad m_1 = im_2, \quad k_1 = -ik_2, \quad k_0 = k_3 \ .$$

Variant (20)

$$G = \begin{vmatrix} 0 & 2k_1 & n_0 + n_3 & n_1 - in_2 \\ 0 & 2k_0 & n_1 + in_2 & n_0 - n_3 \\ 0 & 0 & 0 & 0 \\ 0 & 2l_0 & 2m_1 & 2m_0 \end{vmatrix},$$
$$l_1 = 0, \quad l_2 = 0, \quad l_0 = -l_3 \ ,$$
$$m_0 = -m_3, \quad m_1 = im_2, \quad k_1 = -ik_2, \quad k_0 = -k_3 \ .$$

Variant (21)

$$G = \begin{vmatrix} 2k_0 & 0 & n_0 + n_3 & n_1 - in_2 \\ 2k_1 & 0 & n_1 + in_2 & n_0 - n_3 \\ 0 & 0 & 0 & 0 \\ 2l_1 & 0 & 2m_1 & 2m_0 \end{vmatrix},$$
$$l_0 = 0, \quad l_3 = 0, \quad l_1 = il_2 \ ,$$
$$m_0 = -m_3, \quad m_1 = im_2, \quad k_1 = ik_2, \quad k_0 = k_3 \ .$$

Variant (22)

$$
G = \begin{vmatrix} k_0 + k_3 & k_1 - ik_2 & 0 & 2n_1 \\ k_1 + ik_2 & k_0 - k_3 & 0 & 2n_0 \\ 0 & 0 & 0 & 0 \\ 2l_1 & 2l_0 & 0 & 2m_0 \end{vmatrix},
$$

$$
m_1 = 0, \quad m_2 = 0, \quad m_0 = -m_3 \, ,
$$

$$
n_0 = -n_3, \quad n_1 = -in_2, \quad l_1 = il_2, \quad l_0 = -l_3 \, .
$$

Variant (23)

$$
G = \begin{vmatrix} k_0 + k_3 & k_1 - ik_2 & 2n_0 & 0 \\ k_1 + ik_2 & k_0 - k_3 & 2n_1 & 0 \\ 0 & 0 & 0 & 0 \\ 2l_1 & 2l_0 & 2m_1 & 0 \end{vmatrix}
$$

$$
m_0 = 0, \quad m_3 = 0, \quad m_1 = im_2 \, ,
$$

$$
n_0 = n_3, \quad n_1 = in_2, \quad l_1 = il_2, \quad l_0 = -l_3 \, .
$$

Variant (30)

$$
G = \begin{vmatrix} 0 & 2k_1 & n_0 + n_3 & n_1 - in_2 \\ 0 & 2k_0 & n_1 + in_2 & n_0 - n_3 \\ 0 & 2l_1 & 2m_0 & 2m_1 \\ 0 & 0 & 0 & 0 \end{vmatrix},
$$

$$
l_0 = 0, \quad l_3 = 0, \quad l_1 = -il_2 \, ,
$$

$$
k_0 = -k_3, \quad k_1 = -ik_2, \quad m_1 = -im_2, \quad m_0 = m_3 \, .
$$

Variant (31)

$$
G = \begin{vmatrix} 2k_0 & 0 & n_0 + n_3 & n_1 - in_2 \\ 2k_1 & 0 & n_1 + in_2 & n_0 - n_3 \\ 2l_0 & 0 & 2m_0 & 2m_1 \\ 0 & 0 & 0 & 0 \end{vmatrix},
$$

$$
l_1 = 0, \quad l_2 = 0, \quad l_0 = l_3 \, ,
$$

$$
k_0 = k_3, \quad k_1 = ik_2, \quad m_1 = -im_2, \quad m_0 = m_3 \, .
$$

Variant (32)

$$
G = \begin{vmatrix} k_0 + k_3 & k_1 - ik_2 & 0 & 2n_1 \\ k_1 + ik_2 & k_0 - k_3 & 0 & 2n_0 \\ 2l_0 & 2l_1 & 0 & 2m_1 \\ 0 & 0 & 0 & 0 \end{vmatrix},
$$

$$
m_0 = 0, \quad m_3 = 0, \quad m_1 = -im_2 \, ,
$$

$$
l_0 = l_3, \quad l_1 = -il_2, \quad n_1 = -in_2, \quad n_0 = -n_3 \, .
$$

Variant (33)

$$G = \begin{vmatrix} k_0 + k_3 & k_1 - ik_2 & 2n_0 & 0 \\ k_1 + ik_2 & k_0 - k_3 & 2n_1 & 0 \\ 2l_0 & 2l_1 & 2m_0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$m_1 = 0, \quad m_2 = 0, \quad m_0 = m_3 \,,$$

$$l_0 = l_3, \quad l_1 = -il_2, \quad n_1 = in_2, \quad n_0 = n_3 \,.$$

Conclusion

Let us summarize results. Matrices of the form

$$G = \begin{vmatrix} k_0 + \mathbf{k}\,\vec{\sigma} & n_0 + \mathbf{n}\,\vec{\sigma} \\ & \\ l_0 + \mathbf{l}\,\vec{\sigma} & m_0 + \mathbf{m}\,\vec{\sigma} \end{vmatrix},$$

are described by their determinant [4]

$$\det G = (kk)\,(mm) + (nn)\,(ll) - 2\,(kn)\,(ml) - \\ - 2\,(\,-k_0\,\mathbf{n} + n_0\,\mathbf{k} + i\,\mathbf{k} \times \mathbf{n}\,)\,(\,-m_0\,\mathbf{l} + l_0\,\mathbf{m} + i\,\mathbf{m} \times \mathbf{l}\,)\,; \quad (41)$$

classification of degenerate 4-dimensional matrices, $\det G = 0$, of the rank 1, 2, 3 is elaborated:

$$(n) \longrightarrow 7\,, \quad (m) \longrightarrow 7\,, \quad (n) \longrightarrow 4\,, \quad (l) \longrightarrow 4\,,$$

$$(km) \longrightarrow 5\,, \quad (l,n) \longrightarrow 2\,, \quad (k,n) \longrightarrow 2\,,$$

$$(k,l) \longrightarrow 2\,, \quad (n,m) \longrightarrow 2\,, \quad (m,l) \longrightarrow 2\,,$$

$$(k,m,n) \longrightarrow 2\,, \quad (k,m,l) \longrightarrow 2\,,$$

$$(n,l,k) \longrightarrow 2\,, \quad (n,l,m) \longrightarrow 2\,.$$

Authors are grateful to participants of the seminar of Laboratory of theoretical physics, Institute of Physics of National Academy of Sciences of Belarus for stimulating discussion.

We wish to thank the Organizers of the XLVIII All-Russia conference on problems in Particle Physics, Plasma Physics, Condensed Matter, and Optoelectronics Russia, Moscow, 15–18 May 2012, dedicated to the 100-th anniversary of Professor Ya. P. Terletsky, for opportunity to give a talk on the subject.

References

1. *Red'kov V. M.* Lorentz Group and Polarization of the Light // Advances in Applied Clifford Algebras. — 2011. — Vol. 21. — Pp. 203–220.
2. *Bogush A. A., Red'kov V. M.* On Unique Parametrization of the Linear Group GL(4.C) and its Subgroups by Using the Dirac Algebra Basis // Nonlinear Phenomena in Complecs Sistem. — 2008. — Vol. 11. — Pp. 1–24.
3. *Red'kov V. M., Bogush A. A., Tokarevskaya N. G.* On Parametrization of the Linear GL(4,C) and Unitary SU(4) Groups in Terms of Dirac Matrices // SIGMA. — 2008. — Vol. 4. — P. 021.
4. *Red'kov V. M., Bogush A. A., Tokarevskaya N. G.* 4 × 4 Matrices in Dirac Parametrization: Inversion Problem and Determinant. — arXiv:0709.3574v2. — 2008.

УДК 512.64, 535.1, 535-4

Вырожденные 4-мерные матрицы со структурой полугрупп и поляризационная оптика

Е. М. Овсиюк*, В. М. Редьков†

* *Мозырский государственный педагогический университет им. И.П. Шамякина ул. Студенческая, 28, г. Мозырь, Гомельская обл., 247760, Республика Беларусь*

† *ГНУ «Институт физики им. Б.И Степанова Национальной академии наук Беларуси» пр.Независимости, 68, г.Минск, 220072 Беларусь*

В поляризационной оптике важную роль играют матрицы Мюллера — вещественные 4-мерные матрицы, описывающие воздействие оптических элементов на состояние поляризации света в 4-мерном формализме векторов Стокса. Насущной проблемой является классификация всех возможных классов матриц Мюллера. В частности, специального интереса заслуживают вырожденные матрицы Мюллера с нулевым определителем. В этом контексте, в работе с использованием параметризации 4-мерных матриц на основе базиса матриц Дирака получена классификация простых возможных классов вырожденных матриц Мюллера со структурой полугрупп рангов 1, 2, 3. Метод исследования основан на наложении линейных ограничений на 16 дираковских параметров 4-мерных матриц, при этом требуется совместимость таких ограничений с групповым законом матричного умножения.

Ключевые слова: поляризация света, вырожденные матрицы Мюллера, классификация.