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Estimation of the Relativistic Phase-Shift Formula Applicability to Jupiter's Satellite System

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In this work an estimate of the relativistic phase shift of space body satellite rotation observed from a remote planet is compared with the classical perturbation of the satellite orbit by other space bodies. The calculations are exemplified by Jupiter's satellites. A satellite of the Amalthea group interacting with the Galilean satellites is chosen. The interaction of this satellite with the rest of its group is negligible as compared to that with external satellites, since the mass of any internal satellite is much less than that of external ones.

A gravitational interaction of Jupiter's satellite system has been considered within the weak-interaction approximation for inner satellites neglecting Galilean satellites' action on the phase. Jupiter's system is chosen since it has many satellites whose mutual interaction is rather strong due to small distances between them and their large mass, besides Jupiter is rather close to us, so it is possible to observe directly its satellites in a telescope and to check data empirically.

A gravitational deviation of the chosen inner satellite is calculated to match against the value obtained from the relativistic phase shift formula. The relativistic shift between real and observable phases is given by a formula obtained by A. P. Yefremov in the framework of Quaternion theory. The formula for correction to the phase is a relativistic effect of time delay. The classical correction is estimated using celestial mechanics. An effect of the Galilean satellites on the inner satellites is considered. The phase correction is compared with the value predicted by Quaternion theory of relativity.

In conclusion applicability of this formula has been discussed.

Key words and phrases: relativistic effects, phase shift, Jupiter 's satellite system, Amalthea, quaternionic relativity, formulae estimation.

1. Introduction

There exists a problem of determining space body coordinates due to a plethora of effects on the measurement process. They are: light ray deflection by gravity sources, time delay of the objects moving relative to the observer etc. Thus, it is important to estimate a relative contribution of each effect to the overall picture of measurements and to separate some effects from the others.

This work is devoted to estimation of the relativistic effect illustrated by Jupiter's satellites. Relativistic formula obtained by A. P. Yefremov [1].

Section 2 includes description of the algebra of quaternions and biquaternions and the main notions of General relativity in terms of hypercomplex number algebra. In section 3 a formula for estimating the phase shift is derived. in Section 4 the action of the gravitational field of the Galilean satellites on a chosen inner satellite is calculated, and the values obtained are compared with the corrections due to a relativistic phase shift. In Section 5 the main consequences of the results obtained are briefly discussed.

2. Quaternions and Fundamentals of Quaternion Special Relativity

Now we shall discuss in detail the fundamentals of the theory to find the final formula for the relativistic shift. The quaternion calculation is based on four units,

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one is a scalar and three are vectors $\mathbf{q_1}$, $\mathbf{q_2}$, $\mathbf{q_3}$, satisfying the multiplication rules [2]:

$$1\mathbf{q_k} = \mathbf{q_k}1 = \mathbf{q_k}; \quad \mathbf{q_k}\mathbf{q_l} = -\delta_{kl} + \varepsilon_{kli}q_j. \tag{1}$$

The vector units are imaginary ($\mathbf{q_k^2} = -1$) and related to each other as follows:

$$\mathbf{q_3} = \mathbf{q_1}\mathbf{q_2} = -\mathbf{q_2}\mathbf{q_1}.$$

A linear combination of the vector units with the real unit gives a quaternion number. All quaternions are commutative and associative with respect to addition and associative but not commutative with respect to multiplication. Hamilton also noticed that the three imaginary units may be regarded as forming a vector of the Cartesian coordinate system.

The rule of multiplication of the quaternion units is form-invariant under the transformations of vector units reads:

$$\mathbf{q_{k'}} = U\mathbf{q_k}U^{-1}, \quad \mathbf{q_{k'}} = O_{k'l}\mathbf{q_l} \tag{2}$$

with a full set of the operators U and O of groups $SL(2, C)$ and $SO(3, C)$ respectively, which are isomorphic to the Lorentz group in relativity theory. The transformations (2) give a new set of vectors $\mathbf{q_k}$ satisfying the rule (1), but become a function of the transformation parameter, whereas the real unit remains invariable.

If the quaternion parameters are complex numbers, such an object is called a **biquaternion**:

$$\mathbf{s} = (a_k + ib_k)\mathbf{q_k} = a + ib, \tag{3}$$

where a and b are real. The vectors a and b are orthogonal regardless of the coordinate choice

$$a'_k b'_k = a_n O'_{nk} b_m O'_{mk} = \delta_{mn} a_n b_m = 0.$$

The vector b is aligned with the vector $\mathbf{q_1}$, whereas the vector a is orthogonal to them and aligned with $\mathbf{q_2}$

$$\mathbf{s} = ib_1\mathbf{q_1} + a_2\mathbf{q_2}. \tag{4}$$

This form of representing the BQ number is convenient for considering the physical problems related to relativistic motions.

In Einstein's theory of relativity the interval is given by the formula: $\mathrm{d}s_2 = \mathrm{d}t_2 - \mathrm{d}r_2$ In BQ terms the interval can be written in the form

$$\mathrm{d}\mathbf{s} = (ie_k\mathrm{d}t + \mathrm{d}x_k)\mathbf{q_k}, \tag{5}$$

where the displacement of an observable object $\mathrm{d}x_k$ is orthogonal to the unit vector e_k, showing a direction of time change $\mathrm{d}t$: $e_k\mathrm{d}x_k = 0$.

Let the Σ' system be obtained from Σ by rotation around the axis $\mathbf{q_3}$ by the angle $i\ \psi$:

$$\Sigma' = O_3^{\mathrm{i}\psi}\Sigma, \quad O_3^{\mathrm{i}\psi} = \begin{pmatrix} \mathrm{ch}\,\psi & i\cdot\mathrm{sh}\,\psi & 0 \\ -i\cdot\mathrm{sh}\,\psi & \mathrm{ch}\,\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

Physically, this is an ordinary boost, obviously keeping the BQ vector form-invariant and also giving the coordinate transformation as follows $\mathrm{d}t' = \mathrm{d}t\,\mathrm{ch}\,\psi + \mathrm{d}r\,\mathrm{sh}\,\psi$, $\mathrm{d}r' = \mathrm{d}t\,\mathrm{sh}\,\psi + \mathrm{d}r\,\mathrm{ch}\,\psi$, resulting in the effects of length reduction and time dilation. If the body is at rest in some reference frame, its velocity will be equal to $V = \mathrm{d}r/\mathrm{d}t = \mathrm{th}\,\psi$ in another one.

3. Phase Shift Formula

Now consider a satellite of the planet being observed from the Earth. Let $\tilde{\Sigma} = (\mathbf{q_{\tilde{1}}}, \mathbf{q_{\tilde{2}}}, \mathbf{q_{\tilde{3}}})$ be a fixed coordinate system associated with the Sun, the vectors $\mathbf{q_{\tilde{2}}}, \mathbf{q_{\tilde{3}}}$

form an ecliptic plane, and $\Sigma = (\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3})$ be also a fixed system, connected with the Earth, whose basis vectors are parallel to $\tilde{\Sigma}$. Let the Earth Σ and the observable planet $\Sigma^{'}$ orbit in circles with radii R_E and R_P, and constant linear velocities V_E and V_P. The angular frequency of each rotation is given by the formula $\Omega = V/R$, $\alpha = \Omega_E t$ and $\beta = \Omega_P t$ are the angles between $\mathbf{q_{\tilde{2}}}$ and the direction to the Earth and the planet respectively. All values are measured by the observer on Earth. The coordinates of the planet for the observer:

$$x_2 = R_P \cos\beta - R_E \cos\alpha, \tag{7a}$$

$$x_3 = R_P \sin\beta - R_E \sin\alpha. \tag{7b}$$

The system Σ' is related to Σ by the formula $\Sigma' = O_3^{\mathrm{i}\psi}\Sigma$, and the relative velocity $\Sigma' - \Sigma$, is obtained using (7),

$$V^2 = \left(\frac{\mathrm{d}x_2}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x_3}{\mathrm{d}t}\right)^2 = V_E^2 + V_P^2 - 2V_E V_P \cos(\alpha - \beta) = th^2\Psi$$

defines a hyperbolic parameter $\Psi \ll 1$. The corresponding BQ vector has the form:

$$\mathrm{d}s = \mathrm{d}t' p_{1'} = \mathrm{d}t\left(p_1 + \frac{\mathrm{d}x_2}{\mathrm{d}t}q_2 + \frac{\mathrm{d}x_3}{\mathrm{d}t}q_3\right),$$

giving the correlation time $\Sigma' - \Sigma$, $\mathrm{d}t = \mathrm{d}t' \,\mathrm{ch}\,\psi$. The observable frequency is less than the real one:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{T' \,\mathrm{ch}\,\Psi} = \frac{\omega'}{\mathrm{ch}\,\omega}.$$

Hence, expressing a difference between the observable and theoretical values, it is possible to find:

$$\Delta\varphi = \varphi_r - \varphi_p - \frac{V_E^2 + V_P^2}{2c^2}\omega' t, \tag{8}$$

where φ_r is the observed phase, φ_p the real phase, V_E is the observer velocity , V_P the observed system velocity, ω' is the angular velocity of rotation in the observed system, t the observation time.

4. Estimation of Phase Perturbations

Now consider the system of Jupiter's satellites, i.e. Amalthea's and Galilean groups. The Amalthea group satellites are closest to Jupiter. They move faster than the Galilean satellites having smaller masses, so that the relativistic effects are more significant for them, and their interaction with one another is negligible as compared to the Galilean ones. Therefore, one satellite of the given group is chosen for consideration.

Thus, the system consists of Jupiter, one observed satellite of Amalthea's group and four Galilean ones. The eccentricities of all satellites and deviations from the ecliptic plane are small, hence we consider a flat circular motion. The consideration is aimed at finding a deviation of the motion phase due to gravitational interaction with the four others and comparing the result obtained with the relativistic correction.

Designate Jupiter's mass and that of the satellite under consideration as $\mathbf{M_J}$ and $\mathbf{m}$ respectively, the masses of the Galilean satellites as $\mathbf{M_i}$ where the number of the latter i varies from one to four. Designate the radii of orbits and the phase of an internal satellite and the Galilean ones as r, ϕ, R_i, Φ_i respectively.

First write down the Lagrange function of the system in polar coordinates as follows:

$$L(r,\dot r,R,\dot R,\varphi,\dot\varphi,\Phi,\dot\Phi)=\frac{m}{2}(\dot r^2+r^2\dot\varphi^2)+\frac{GM_Jm}{r}+$$
$$+\sum_{i=1}^{4}\frac{M_i}{2}(\dot R_i^2+R_i^2\dot\Phi_i^2)+\sum_{i=1}^{4}\frac{GM_JM_i}{R_i}+\frac{1}{2}\sum_{i=1}^{4}\sum_{j}^{4}\frac{GM_iM_j}{D_{ij}}+\sum_{i=1}^{4}\frac{GmM_i}{D_i},$$

where

$$\begin{cases} D_i=\sqrt{R_i^2+r^2-2R_ir\cos(|\Phi_i-\varphi|)},\\ D_{ij}=\sqrt{R_i^2+R_j^2-2R_iR_j\cos(|\Phi_i-\Phi_j|)}\end{cases}$$

are the distances between the chosen satellite and the Galilean one, and between the Galilean satellite in pairs respectively.

The first two terms are the Lagrange function of the satellite in the zero approximation, they are its kinetic and minus potential energy relative to Jupiter respectively; the next two terms is the same for the Galilean satellites, the next term is an interaction between the Galilean satellites, and the last term is an interaction between internal and external satellites.

Then, substituting this Lagrange function into the Lagrange equations [3]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot q_i}-\frac{\partial L}{\partial q_i}=0,$$

(where q — generalized coordinate) and assuming orbits to be circular, after some transformations we obtain the system:

$$\begin{cases} \ddot\varphi=-\displaystyle\sum_{i=1}^{4}\frac{GM_iR_i}{D_i^3r}\sin(|\Phi_i-\varphi|),\\ r\dot\varphi^2=\displaystyle\frac{GM_J}{r^2}-\sum_{i=1}^{4}\frac{GM_i}{D_i^3}(R_i\cos(|\Phi_i-\varphi|)-r),\\ \ddot\Phi_i=-\displaystyle\frac{Gmr}{R_iD_i^3}\sin(|\Phi_i-\varphi|)-\frac{G}{R_i}\sum_{j=1}^{4}\frac{M_jR_j}{D_{ij}^3}\sin(|\Phi_i-\Phi_j|),\\ R_i\Phi_i^2=\displaystyle\frac{GM_J}{R_i^2}-\frac{Gm}{D_i^3}(r\cos(|\Phi_i-\varphi|)-R_i)+G\sum_{j=1}^{4}\frac{M_j}{D_{iij}^3}(R_i-R_j\sin(|\Phi_i-\Phi_j|)).\end{cases}$$

We have obtained a system of second-order differential equations of five variables. Naturally, the solution in a general form is very cumbersome. Hence we recourse to the following method. Take the same system, but only for two satellites, calculate deviations for each Galilean satellite separately and then sum up deviations. For this purpose, in each sum only one index i should be retained, other satellites are considered infinitely remote. The summation index is omitted. The system takes the form:

$$\ddot\varphi=-\frac{GMR}{D^3r}\sin(|\Phi-\varphi|),\tag{9a}$$

$$r\dot\varphi^2=\frac{GM_J}{r^2}-\frac{GM}{D^3}(R\cos(|\Phi-\varphi|)-r),\tag{9b}$$

$$\ddot\Phi_i=-\frac{Gmr}{RD^3}\sin(|\Phi_i-\varphi|),\tag{9c}$$

$$R\Phi^2=\frac{GM_J}{R^2}-\frac{Gm}{D^3}(r\cos(|\Phi-\varphi|)-R).\tag{9d}$$

If one considers equations (9b) and (9d), neglecting interaction between the Galilean satellites and assuming that $\dot{\varphi} = \omega = 2\pi/T$, it is possible to obtain Kepler's third law:

$$\frac{r^3}{t^2} = \frac{R^3}{T^2} = \frac{GM^j}{4\pi^2} = \text{const},$$

where t and T are the orbital periods of the internal and external satellite respectively.

Subtracting equation (9a) from (9c), we obtain:

$$\ddot{\Phi}_i - \ddot{\varphi} = \frac{GMR}{D^3 r}\sin(|\Phi - \varphi|) - \frac{Gmr}{RD^3}\sin(|\Phi_i - \varphi|) = \frac{G}{D^3}\sin(|\Phi - \varphi|)\left(\frac{MR}{r} - \frac{mr}{R}\right).$$

Since $M \gg m$ and $R \gg r$, the second term may be neglected. Introducing the substitution $\Phi - \varphi = x$, we obtain the equation:

$$\ddot{x} = \frac{GMR}{rD^3}\sin(x),$$

Integrating it, we have:

$$\dot{x}^2 = C - \frac{2GM}{r^2\sqrt{R_i^2 + r^2 - 2R_i r\cos(|\Phi_i - \varphi|)}}. \tag{10}$$

Clarify the meaning of a constant C. When phases of motion coincide, the second satellite does not act on the first one, and we have the first approach. Now return to the old variables $\dot{x} = \dot{\Phi} - \dot{\varphi} = \Omega_0 - \omega_0$ and write down the frequencies following the classical gravity theory as

$$\omega_0^2 = \frac{GM_j}{r^3}, \quad \Omega_0^2 = \frac{GM_j}{R^3}. \tag{11}$$

Substituting it in (10), we obtain

$$C = \left(\sqrt{\frac{GM_j}{R^3}} - \sqrt{\frac{GM_j}{r^3}}\right)^2 + \frac{2GM}{r^2(R - r)}.$$

Substituting it in (10), expressing frequencies from (11) and taking into account that the angular frequency of an internal satellite much exceeds that of an external one, we obtain:

$$\Delta\omega = \Omega - \omega = \sqrt{\omega_0^2 + \frac{2GM}{r^2(R - r)} - \frac{2GM}{r^2\sqrt{R_i^2 + r^2 - 2R_i r\cos\left(|\Phi_i - \varphi|\right)}}} - \omega_0.$$

Expanding the root in series, after some transformations we obtain the formula for correcting the rotation phase of an internal satellite for attraction to one of the external satellites:

$$\Delta\omega = M\sqrt{\frac{G}{rM_J}}\left(\frac{1}{R - r} - \frac{1}{\sqrt{R_i^2 + r^2 - 2R_i r\cos\left(|\Phi_i - \varphi|\right)}}\right). \tag{12}$$

Consider some extreme cases of the function (12). As an internal satellite Methis of the Amalthea's group is chosen, and Io as an external one.

1. $\Phi - \varphi = 0, \quad \Delta\omega = 0$;
2. $\Phi - \varphi = \pi/2, \quad \Delta\omega = 1.6 \cdot 10^{-9}$ rad/sec;

3. $\Phi - \varphi = \pi$, $\Delta\omega = 2.3 \cdot 10^{-9}$ rad/sec.

The plot of the dependence $\Delta\omega$ on the phase difference $\Phi - \varphi$ built in Maple's system is shown on Fig. 1, where the phase difference in radians is given along the horizontal axis, and the phase shift — in inverse centimetres along the vertical one.

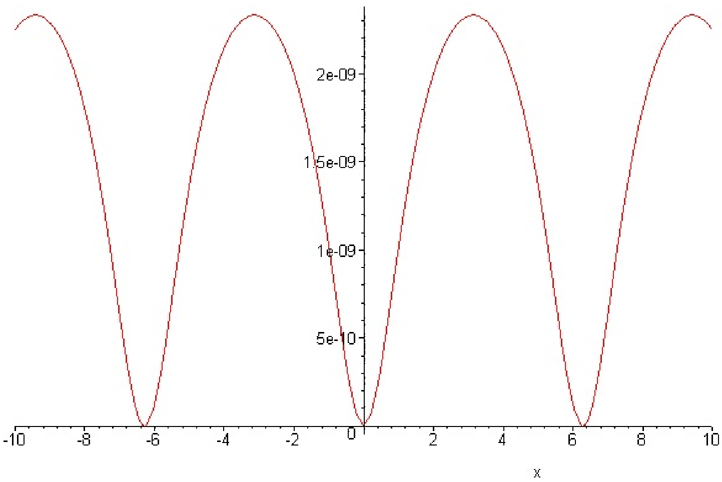

Figure 1. Dependence of the phase shift of difference of phases

In Yefremov's article [4] a shift of the Methis phase is obtained, which amounts to 15.77 seconds for 100 years. Conversing it in a relative angular acceleration, we obtain $2.5 \cdot 10^{-24}$ rad/sec, which is by five orders less than the classical one.

Estimate the trajectory section for which it is possible to separate a relativistic effect from a classical one. For this purpose, we expand $\Delta\omega$ near zero more precisely and obtain the formula:

$$\Delta\omega = \sqrt{(\Omega_0 - \omega_0)^2 + \frac{GMRx^2}{r^3(R-r)^3}} - (\Omega_0 - \omega_0).$$

Solving it, we obtain an angle for which the angular frequency variations due to a relativistic motion are commensurable or exceed those for gravitational interaction

$$\varphi = \left(-2.3 \cdot 10^{-2}, 2.3 \cdot 10^{-2}\right)'$$

and the orbit section length satisfying the condition $\Delta l = 52.5$ km.

5. Conclusion

In the present paper the action of a relativistic effect of the phase shift on the measurement of Jupiter's satellite has been estimated.

A brief derivation of the classical correction to the motion of the satellite Methis due to gravitational interaction with the Galilean satellite Io is presented. The maximal classical correction proved to more than five orders exceed the relativistic one, thus the latter may be neglected if a high accuracy in calculating the satellite motion is not required.

The distance at which the effect is significant is about the satellite's diameter, too small to be observable for Jupiter's satellites, that is, it is possible to consider the system of Jupiter's satellites to be nonrelativistic.

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Оценка области применимости релятивистских поправок на примере движения внутренних спутников Юпитера

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В данной работе даётся оценка эффекта релятивистского сдвига фазы вращения спутника космического тела при наблюдении с отдалённой планеты в сравнении с классическим возмущением орбиты спутника другими космическими телами. Вычисления ведутся на примере спутников Юпитера. Выбирается один спутник из группы Амальтеи, который взаимодействует со спутниками Галилея. Взаимодействием данного спутника с остальными из его группы можно пренебречь по сравнению с взаимодействием с внешними спутниками, так как масса внутренних много меньше.

Гравитационное взаимодействие спутников Юпитера рассматривается в предположении, что взаимодействие внутренних спутников мало, а внешние спутники не влияют на изменение своих фаз. Система Юпитера выбрана из-за того, что он имеет довольно большую систему спутников, взаимодействие между которыми за счёт малого расстояния и большой массы довольно сильное, и система Юпитера достаточно близка к нам для прямых измерений и все данные проверить эмпирически.

Вычисляется гравитационная поправка выбранного внутреннего спутника и сравнивается со значением сдвига фазы, полученного с помощью формулы для релятивистского сдвига фазы.

Релятивистский сдвиг между реальной фазой и наблюдаемой даётся формулой, полученной А. П. Ефремовым в развиваемой им кватернионной теории относительности. Формула поправки к фазе по сути представляет собой релятивистский эффект замедления времени. Классическая поправка оценивается с помощью небесной механики и теории возмущений. Её формула получается путём вычислений через полный лагранжиан взаимодействия спутников Юпитера и самой планеты. Для простоты орбиты считаются круговыми и полученная система рассматривается попарно для выбранного внутреннего спутника с каждым из галилеевых и далее влияние складывается. Полученная система решается, и полученная поправка к фазе сравнивается со значением предсказываемым формулой кватернионной теории относительности.

Делаются выводы об области применимости последней.

Ключевые слова: релятивистские эффекты, сдвиг фазы, система спутников Юпитера, Амальтея, кватернионная теория относительности, проверка формулы.

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